SIMULATION OF LINER OPERATIONS

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'Here sails the vali-
gen sites in Bergen.

I. *Introduction*[1]

Liner operations are characterized by a specific number of ships sailing one predetermined route to several ports at various times, between ports in Europe, for instance, and ports on the West Coast of North America. The ships are usually carrying an assortment of merchandise, shipping space normally being purchased by more than one individual at each port. The shipowner's, or shipping operators' problem is to choose a sequence of ports which will insure a high demand for shipping space to coincide with the liner's time of arrival. Returns to the shipowner will depend on decision parameters such as the number of ships, the ships' characteristics and the schedule chosen for use.

The inflow of goods to each port may vary considerably over time (by seasons, for example). It is quite difficult to adopt a schedule that will take advantage of peak demand while having reasonable utilization of the liner capacity during periods of lower inflows of goods. The addition of extra ships during peak periods is one possible solution to this problem. The problem, however, of deciding the exact number of ships which will sail the route at any one time remains.

If information concerning the inflow of goods to each port can be provided (from past shipping records, for instance) or if fairly reliable forecasts can be made, the possibilities of savings and/or increased earnings exist. Simulation is a useful method of analyzing the optimization process which occurs here. The effects of uncertainty variables such as inflows of goods, waiting times and freight rates can be inspected and evaluated using this method.

[1] The simulation model described was developed on the initiative of Albert Harloff at A.S. Bergens Mekaniske Verksteder. The project has received support from A.S. Bergens Mekaniske Verksteder, Norwegian School of Economics and Business Administration, and the Royal Norwegian Council for Scientific and Industrial Research.

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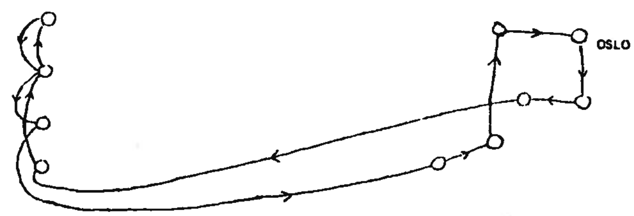


Figure 1.

II. *Survey of the model*[1]

In working with the model we have used a concrete route as an example of what can be done. This route included a number of ports in Europe and a number of ports on the West Coast of North America. This route is illustrated in Figure 1.

Thus we have the ports clustered in two geographical areas, where the distance between the two areas is large in comparison to the distances between ports within the areas. On the figure is drawn an example of an arrival sequence. The choice of this sequence can obviously be crucial for the result of the operation of the route. If we look at the ports on the West Coast of North America, we see that the ships will perhaps call at each port only once, either on their way from Europe or on their way back. It may also be profitable to call at some ports to unload on the way from Europe, and call at the same ports to load on the way back, in order to make better use of the liner's capacity. The choice, of course, will depend on a comparison of the extra revenues obtainable from increased quantities of cargo, and the extra costs, such as fixed costs incurred when ports are called at twice instead of once. In the model this event may be examined by varying the sequence in which the ports are called. For the case where a port is called at twice during one roundtrip, the model will automatically assure that cargoes are unloaded at the port on the way from Europe and loaded on the way back.

The inflow of goods to the ports will determine the optimal number of ships, and, as a consequence, the sailing intervals. The type of goods available to be shipped will determine if it will be profitable to use ships with special holds, for instance holds designed for goods which need cold storage. We have a

[1] A preliminary description can be found in Doksrød et al. (4).

number of factors then, which exert influence on each other. The optimal number of ships can be different for different arrival sequences, and at the same time the optimal number can vary with the ship type. Our model ought to incorporate a simultaneous adjustment of arrival sequence, the number of ships, and the ships' characteristics (i.e., speed, deadweight, total volume, volumes for special holds, and equipment for loading and unloading).

The interrelations mean that we must content ourselves with some type of suboptimization. In the model this is done in the following way: We start out with a certain ship-type. We then choose in advance the arrival sequences that seem most usable and that we want to compare. Within each arrival sequence we carry out as many simulations as we want with various numbers of ships (one simulation corresponds to one roundtrip with each ship), and for each arrival sequence we record the optimal number of ships. Finally, we will have found which of the arrival sequences inspected is the optimal, and also an optimal number of ships for this sequence.

We can then alter the data for the ships and repeat the operation described above. This can be done for as many types of ships as desired; in this way we will also get an indication of which type of ship is the optimal.

The criterion used for the optimization is net profit per day with time horizon equal to the time required for one roundtrip. The roundtrip time follows from the optimization. It will vary with quantities and types of cargo, because different goods need different loading and unloading times. What is being done, is therefore to maximize the net profit per day for each roundtrip at a time. In order to do this optimization, we must have a starting port where the ships are always initially empty. The time used for one roundtrip is then calculated from the point of time when the ship starts loading in the starting port and ending when it starts loading in the same port after having called at the other ports in the route (in our example we use Oslo as a starting port; see Figure 1). This arrangement may be unrealistic for some routes. Relaxing this restriction will generally leave us without an exact optimum; the degree of approximation however, can be good if we choose as the starting port the port with lowest average amount of goods in transit. This approximation becomes better if the goods which remain in the liner at this port add considerably to the profit, so that there is usually no doubt that they should be carried.

During the simulations we generate stochastic inflows of goods to the ports on the basis of data for expectations and variances. The generating process for each simulation occurs, for all of the goods, before the ship starts loading at the starting port. Then, before each roundtrip, the model chooses the types and quantities of goods that will maximize net profit per day.

Waiting times in the ports may vary considerably. These times depend on the time, on the day and on the week, during which the ship arrives at the port. Waiting times also depend on conditions in the ports. These waiting times can be generated as stochastic variables for each port based on data for means and variances of the actual waiting times.

III. *The maximization part of the model*[1]

The notations used in the model are the following:

Service speed	s	(knots)
Distance sailed per roundtrip	d	(miles)
Carrying capacity (deadweight)	K	(tons)
Total volume in cubic feet	V	(cf)
Other constraints on volume	$V_1, V_2,$—	(cf)
Fixed costs for the ships	C_1	(\$/day)
Additional costs at sea (fuel etc.)	C_2	(\$/day)
Fixed port costs in port no i	H_i	(\$/call)
Loading costs for cargo no. j	k_j	(\$/ton)
Unloading costs for cargo no. j	k_j'	(\$/ton)
Loading speed for cargo no. j	l_j	(tons/day)
Unloading speed for cargo no. j	l_j'	(tons/day)
Stowage factor for cargo no. j	q_j	(cf/ton)
Freight rate for cargo no. j	f_j	(\$/ton)
Waiting time at port no. i (can be generated as a stochastic variable)	v_i	(days)
Number of types of cargoes	n	
Number of ports in the arrival sequence	m	
Amount of cargo no. j available	w_j	(tons)
Amount of cargo no. j to be carried (the unknowns)	x_j	(tons)

[1] A model similar to this part of our model is discussed in Pollak, Novaes and Frankel [6].

By a cargo we mean one type of good which is carried from one specific port to another specific port. Information on port of origin and port of destination for each cargo is initially read into the model. If one type of cargo is carried from, or eventually to, several different ports, this cargo will get a new index j for each port of origin and/or port of destination.

By fixed costs we will mean the sum of all costs which do not depend on whether the ship sails or not. These may contain depreciation, crew wages, stores and supplies, insurance (excluding cargo) etc. We assume that these may be distributed over time with a fixed charge per day.

Using the notation above we can set up the following expressions:

Time at sea + waiting times (i.e. fixed time per roundtrip independent of the amount of cargo):

$$\beta_0 = \frac{d}{24 \cdot s} + \sum_{i=1}^{m} v_i$$

For simplicity we will write $\sum_{i=1}^{m}$ which is to be understood as meaning that i runs over all the ports in the arrival sequence considered, and, for the case where we have double calls at some port during a roundtrip, one port will correspond to two i's in the summation above.

Loading and unloading time for cargo j:

$$\beta_j = \frac{1}{l_j} + \frac{1}{l_j'} \qquad j = 1, \ldots, n.$$

β_0 is measured in days, $\beta_j, j = 1, \ldots, n$ in days/ton.

Fixed costs (with negative sign) independent of the amount of cargo:

$$\alpha_0 = -(C_1 + C_2)\frac{d}{24 \cdot s} - \sum_{i=1}^{m} H_i - C_1 \cdot \sum_{i=1}^{m} v_i =$$

$$= -C_1 \cdot \beta_0 - C_2 \cdot \frac{d}{24 \cdot s} - \sum_{i=1}^{m} H_i$$

Value added per ton for cargo j:

$$\alpha_j = f_j - k_j - k_j' - C_1\left\{\frac{1}{l_j} + \frac{1}{l_j'}\right\} = f_j - k_j - k_j' - C_1 \cdot \beta_j$$

$$j = 1, \ldots, n$$

Value added is here defined for our purpose, in that we also subtract from the revenue the fixed costs per day which accrue during the time needed for loading and unloading of one ton of cargo j.

We wish to maximize the following expression:

$$P = \frac{\alpha_0 + \alpha_1 x_1 + \ \dots \ + \alpha_n x_n}{\beta_0 + \beta_1 x_1 + \ \dots \ + \beta_n x_n}$$

where the α_j's and β_j's are the coefficients and α_0 and β_0 the constants defined above. P can now be maximized using parametric linear programming[1].

In order to be able to write the objective function in the form of P above, we first have to make some simplifications. We disregard non-linearities in connection with loading and unloading. Since work is usually done only at certain times of the day, overtime work requiring extra pay, loading and unloading speed, and thus loading and unloading cost, can vary depending on what time of the day the ship arrives at the port. In order to apply the above method, we calculate a certain number of hours of work per day in each port, i.e. a certain amount of each type of cargo can be loaded or unloaded per day.

Among the costs which are calculated as fixed independent of the amount of cargo, the fuel cost is obviously not fixed. It is however difficult to treat it in any other fashion. This cost will increase with increasing amount of cargo although not proportionally. A part of this cost should therefore be subtracted from the value added for each cargo, varying according to the weight of the cargo, how far it is to be carried, and how loaded the ship is in advance. This consideration could influence the choice of optimal cargoes. Tests show, however, that in actual practice the numerical differences are so small that they very seldom would influence the choice of cargoes, and then only the choice of those cargoes whose freight rates and other characteristics make them marginal (i.e. they are close to the limit which would prevent them from being loaded). This fact means that for practical purposes, the resulting costs are nearly equal for the two or more cargoes being considered at that time.

[1] See Appendix A.

Another difficulty is that the total cost, and therefore the net profit, can differ considerably from the correct amount. This will occur if one calculates the fuel cost as a fixed amount per day, i.e. for a fixed weight of cargo, instead of calculating it separately between each pair of ports. This difference is particularly important if the volume becomes the overriding constraint. Elimination of this effect can be accomplished by reading into the model information about fuel cost per day for various deadweights and then interpolating between these values to find the approximately correct cost.

The constraints may be thought of as being built up from several blocks.

1) Constraints which assure that the amount of each cargo loaded must be less than or equal to the available amounts of each cargo, i.e.

$$x_i \leq w_i\,, \qquad i=1,\ \ldots\ldots, n$$

where all w_i, $i = 1, \ldots, n$ are stochastic variables generated before each roundtrip.

2) Weight constraints for each port:

$$\underset{\text{loaded}}{\text{All cargoes}} + \underset{\text{in transit}}{\text{all cargoes}} - \underset{\text{unloaded}}{\text{all cargoes}} \leq K$$

or:

$$\sum_{j=1}^{n} a_{n+i,j}x_j \leq K\,, \qquad i=1,\ \ldots, n$$

Here we have $a_{n+i,j}=1$ if cargo j is carried from port i to port $i+1$ (also in transit), and 0 otherwise.

3) Volume constraints for each port:

$$\underset{\text{loaded}}{\text{All cargoes}} + \underset{\text{in transit}}{\text{all cargoes}} - \underset{\text{unloaded}}{\text{all cargoes}} \leq V$$

or:

$$\sum_{j=1}^{n} a_{n+m+i\,j}x_j \leq V\,, \qquad i=1,\ \ldots, m$$

Here $a_{n+m+i,j}=q_j$, i.e. equal to the stowage factor of cargo j if this cargo is carried from port i to port $i+1$, and 0 otherwise.

Further we may for example have:

All cold cargoes + all cold cargoes − all cold cargoes $\leq V_1$
loaded in transit unloaded

or

$$\sum_{j=1}^{n} a_{n+2m+i,j} x_j \leq V_1\,, \qquad i=1,\ldots,m$$

Here $a_{n+2m+i,j} = a_{n+m+i,j}$ if cargo j is cold cargo, and 0 otherwise.

The statement concerning carrying goods from port i to port $i+1$ must be modified for port m. In this case $i+1$ must be replaced by 1, i.e. the starting port.

As an example we may consider five cargoes carried in an arrival order containing five ports. The arrival sequence may be enumerated $1-2-3-4-5-1$.

Cargo no.	Port of origin	Port of destination
1	1	4
2	3	5
3	2	1
4	1	3
5	4	1

Cargo 1 and 3 are cold cargoes.

The information above will in the model produce the following constraints:

$$\begin{array}{lllllllllll}
x_1 & & & & & & & & & \leq w_1 \\
 & & x_2 & & & & & & & \leq w_2 \\
 & & & & x_3 & & & & & \leq w_3 \\
 & & & & & & x_4 & & & \leq w_4 \\
 & & & & & & & & x_5 & \leq w_5 \\
x_1 & & & & & + & x_4 & & & \leq K \\
x_1 & & & + & x_3 & + & x_4 & & & \leq K \\
x_1 & + & x_2 & + & x_3 & & & & & \leq K \\
 & & x_2 & + & x_3 & & & + & x_5 & \leq K \\
 & & & & x_3 & & & + & x_5 & \leq K \\
q_1x_1 & & & & & + & q_4x_4 & & & \leq V \\
q_1x_1 & & & + & q_3x_3 & + & q_4x_4 & & & \leq V \\
q_1x_1 & + & q_2x_2 & + & q_3x_3 & & & & & \leq V
\end{array}$$

$$
\begin{array}{lllll}
 & q_2x_2 & +q_3x_3 & +q_5x_5 & \leq V \\
 & & q_3x_3 & +q_5x_5 & \leq V \\
q_1x_1 & & & & \leq V_1 \\
q_1x_1 & & +q_3x_3 & & \leq V_1 \\
q_1x_1 & & +q_3x_3 & & \leq V_1 \\
 & & q_3x_3 & & \leq V_1 \\
 & & q_3x_3 & & \leq V_1
\end{array}
$$

IV. *Dual evaluators or shadow prices*

From the maximization of a linear programming problem one gets as part of the solution a set of dual evaluators, also called shadow prices, one for each constraint, which indicates how much the objective function (e.g. the profit) would increase per unit increase in the right-hand side of the actual constraint[1]. By summing the respective dual evaluators one can find the result of an increase in several constraints simultaneously. This information can be very useful[2].

In our special problem we are interested in how much net profit per day would increase per ton increase in deadweight and/or per cf. increase in total volume for cold cargoes. The non-linear objective function causes the dual evaluators not to be immediately determinable from the optimal simplex-tableau as is the case with linear profit function; they can, however, easily be calculated[3].

After each simulation we can get some sort of internal prices on the input factors weight and volume. These values can tell us how much we at most should be willing to pay per day for extra units (tons or cf) of each factor. Another example of the use of these values is for comparisons between different types of volume. If, for instance, the dual evaluator for volume of cold cargo is higher than for total volume, the volume of cold cargo should be a larger portion of the total volume.

The dual evaluators give information that we otherwise would have been able to get only by alternative simulations where different combinations of weight and volume were examined. These extra simulations would require considerable amounts of computer time, while calculation of the dual evaluators for each simulation can be done in fractions of a second.

[1] Under the assumption that the basis does not change.
[2] See e.g. A. Charnes and W. W. Cooper [1].
[3] See Appendix A2.

V. *Generating the cargoes*

The modified simplex algorithm which is being used, is only a deterministic method of choosing from given cargoes those amounts which will result in the largest net profit per day per ship. The inflow of cargoes is stochastic and we assume that this inflow can be characterized by the expected values and the variances. These values may be based on historical data or eventually on forecasts. The availability of data is a problem that will not be treated in this article. We will asume that such data can be attained. In the simulation model we may choose between the following distributions for the inflow of cargoes:

a) A deterministic solution based on given amounts of cargoes in the ports.
b) Poisson distribution.
c) Lognormal distribution.

This distribution is non-symmetric with median below the mean. It always give values greater than zero.

d) Uniform distribution.
e) Beta distribution.

The last three probability distributions are generated using a random number generator for random numbers between 0 and 1.

Expected values and variances are given per day. We thus generate stochastic amounts of inflows of goods per day and multiply these by the number of days since the last ship was in the actual port[1].

By reading into the model seasonal indices for each cargo, one will often get a more realistic model.

VI. *The simulations*

The simulations can roughly be described as follows[2]. First we read characteristics of all the cargoes, all the ports and all the ships that we are going to use. We then read the first arrival sequence. For each sequence the simulations are started with

[1] In reality we use an approximation for this period of time which will be described later.

[2] See flow-chart in Appendix B.

a certain number of ships in the route. With this number of ships we do as many simulations as desired (one simulation corresponds to one ship having made one roundtrip). Finally the mean and standard deviation for net profit per day for all the simulations are calculated, and also the mean roundtrip time. From the dual evaluators for each simulation we can compute the average increase in net profit per day resulting from increases in deadweight and/or volume.

The number of ships is now increased by one and the same procedure is repeated, using the same arrival sequence. The new average net profit per day is compared with earlier simulations. If the profit has increased, the number of ships is increased by one. The simulations stop when the average profit is reduced. The optimal number of boats for the current arrival sequence is thus one less than the last number inspected.

We then read new arrival sequences and repeat the same procedure as above. Each time we compare the average net profit for the optimal number of ships with the highest we had reached before. Finally we will have found an optimal arrival sequence, a corresponding optimal number of ships, and the information we get from the dual evaluators for this case. These evaluators can give us some idea of whether the type of ship we are using, is the optimal.

A special problem concerns the goods which are left behind by the ships. In the model we have, more or less arbitrarily, established that cargoes left behind will wait until the next ship comes. If they are left behind once more, then they leave the model (e.g. they are being carried in some other way).

Another problem is the number of days to use as the basis for generating cargo inflows. The correct method would be to compute the number of days from the time the last ship left the port until the next ship leaves the port. The cargo is, however, optimized assuming that we have full information on the amounts of cargoes at each port from the beginning of each roundtrip. The time until each port is left is thus a result of the optimization, while the optimization is influenced by the number of days for which the cargoes are generated. In the model we approach this difficult interaction in the following way: When there is only one boat in the route, the inflow of cargoes is generated for the number of days needed for the preceding roundtrip. Correspondingly we use, when we have more than one ship, the time between arrivals of ships at the starting port. The error made in this way should be negligible.

Certain steps have to be taken in order to get the simulations for each number of ships started. In particular this involves specifying an initial roundtrip time to use as a basis for generating the inflow of goods for the first roundtrip. However, we do not want this specification of initial roundtrip time to have any effect on the results. It is therefore wise to exclude the first one or two simulations in computing mean and standard deviation for net profit.

VII. *Control of scheduling*

When more than one ship is used, the problem arises of how to schedule the ships. Since we use the number of days since the last ship started to load at the starting port as the basis for generating the inflow of cargoes, an accidental variation in the roundtrip time in some direction may cause a ship to start approaching the ship ahead of it. The nearer this ship comes to the one in front, the less cargoes are generated; this will result in the ship approaching the other ship even more. We try to control this difficulty by a clock which registers the current time. The clock for each ship is registered every time the ship is about to start loading in the starting port.

The following variables are used:

Current time for ship no. i (clock)	Cl_i
Time basis for generating cargoes for ship no. i	t_i
Last roundtrip time	T
Upper constraint on roundtrip time for the ship that is about to load in the starting port (ship no. $i+1$)	TU_{i+1}
Corresponding lower constraint	TL_{i+1}
Permitted deviation in „phase" between the ships	D
Number of ships	n

The deviation D will have to vary with the number of ships since the ships will get out of „phase" more easily as their number increases. In the model this is accomplished by dividing an initial D_{in} by the current number of ships. If for example $D_{in} = 10$ days we have:

Number of ships	D
2	5 days
3	3 1/3 days
4	2 1/2 days

After having unloaded ship no. i in the starting port, new values for our variables are computed as follows:

$$
\begin{aligned}
Cl_i &= Cl_i + T \\
t_i &= Cl_i - Cl_{i-1} \\
TU_{i+1} &= Cl_i - Cl_{i+1} + t_i + D \\
TL_{i+1} &= TU_{i+1} - 2D
\end{aligned}
$$

For $i=1$, $i-1$ is replaced by n. For $i=n$, $i+1$ is replaced by 1.

The result of this control will be that a ship may wait for some time extra in the starting port in order not to come too close to the ship ahead of it. This waiting occurs only if there was not enough goods on the last roundtrip to increase the net profit per day when the extra waiting time is taken into account. During simulation this fact is taken care of by using the value TL instead of the sum $\beta_0+\beta_1 x_1+ \ldots.$ in the denominator for the expression of P until this sum exceeds TL.

It may also happen that a ship will have to leave cargo which it otherwise would have carried. This happens if it must have finished unloading in the starting port by a certain point of time in order not to drop too far behind the boat ahead of it. This problem is solved by the additional constraint:

$$\beta_1 x_1 + \beta_2 x_2 + \ \ldots\ldots \ + \beta_n x_n \leq TU - \beta_0 .$$

It is possible that, without too large a loss in reliability, one could make the simplification of always looking at one of the boats and generate the inflow of cargoes for the time this boat uses on each roundtrip. We would then divide the inflow of each cargo by the number of boats. All the boats would have the same composition of cargo, and we would save a great deal of simulation time. Scheduling would, of course, be no problem any longer. The model used at present, however, allows us to incorporate the possibility of letting one boat skip a port where the amount of cargo waiting is small and let the cargo wait for the next boat. Otherwise the ships would have to call at all the ports no matter how little cargo was available.

VIII. *Cargoes with priority*

For some routes the ships may be obliged to always carry some types of cargo even though they could be less profitable than other available cargoes. This will depend on the concession or special agreements. In the model we use a parameter for each cargo which tells us whether the cargo has priority or not. Cargoes with priority will be loaded first; the model will then choose the most profitable cargoes from those remaining using the modified simplex-algorithm. This arrangement means that, on entering the simplex-algorithm, the following changes have taken place with the constraints: $w_i = 0$ for the cargoes with priority. In addition $TU - \beta_0$ for the last constraint is reduced correspondingly.

IX. *Applications of the model*

In the model there are several variables which might be optimized. We have mentioned arrival order and number of ships, but this optimization depends upon the specification of type of ships. One might also be interested in testing the result of alternative types of ships as to deadweight, volume, volume of special holds, loading and unloading equipment, service speed etc.

One can also simulate to find an optimal utilization of ships already sailing. To make this possible, we can in the model use ships with different characteristics simultaneously. In the liner trade one usually has to establish a service such that the ships arrive at rather fixed times and with even intervals. The time schedule for such a service is in the model a by-product of the optimal combination. It is also possible to simulate such that the ships always use a fixed time per roundtrip as determined beforehand; the cargo will then be optimized taking this fact into consideration. Technically the schedule is fixed by putting $D_{tn} = 0$ for control of scheduling.

Finally one might mention the possibility of using the model in the operation of an existing route. Then all the data for the ships will be given. At certain points of the route one may know with certainty the amounts of cargoes in the nearest ports, while the amounts in the rest of the ports are more uncertain. Based on given amounts of some cargoes and probability distributions for the uncertain ones, one could simulate and find which cargoes ought to be loaded; eventually determining if the arrival sequence should be changed somewhat can be made possible.

Since a model is an abstraction of the real world, it, of course, has to be incomplete and inexact in several respects. The real world is far too complex to be analyzed adequately without the simplification that a model provides. The real basis for judging the usefulness of a model is not if it is entirely exact, but rather contains enough sufficiently relevant elements to be efficiently used by managers to improve their results. We believe that this model can be useful in that respect.

In the work with the model we have put emphasize in making it so general that it can be used on various types of problems in liner operations. With the framework now built, one can add special aspects of reality which each particular user might want to take into consideration.

APPENDIX A

In this Appendix we will briefly describe the principle behind the solving of a maximization problem with linear constraints where the objective function is non-linear, but consists of a ratio where both numerator and denominator are linear in the variables[1]. Finally the solution of a numerical example will be calculated. We assume that the reader is familiar with the simplex-method[2].

1) *The primal problem*

Our problem can be formulated as follows[3]:

$$\text{Max}\, P = \frac{\alpha_0 + \alpha_1 x_1 + \; \dots \; + \alpha_n x_n}{\beta_0 + \beta_1 x_1 + \; \dots \; + \beta_n x_n} \tag{1}$$

subject to:

$$\begin{aligned} a_{11}x_1 + \; \dots\dots \; + a_{1n}x_n &\leq b_1 \\ \dots\dots\dots\dots\dots\dots\dots\dots & \\ a_{m1}x_1 + \; \dots\dots \; + x_{mn}x_n &\leq b_m \end{aligned} \tag{2}$$

$$x_i \geq 0, \quad i = 1, \; \dots, n.$$

[1] See also Pollak, Novaes and Frankel [6].
[2] See e.g. Charnes and Cooper [1], Dantzig [2] or Simonnard [7].
[3] This formulation is called fractional programming.

We introduce slackvariables $x_{n+1}, \ldots, x_{n+m}$, and (2) is then equivalent to

$$
\begin{aligned}
a_{11}x_1 + \ldots\ldots + a_{1n}x_n + x_{n+1} &= b_1 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
a_{m1}x_1 + \ldots\ldots + a_{mn}x_n + x_{n+m} &= b_m
\end{aligned}
\tag{3}
$$

$$x_i \geq 0, \quad i = 1, \ldots, n+m$$

The objective function can be written:

$$(a_0 - \beta_0 P) + (a_1 - \beta_1 P)x_1 + \ldots\ldots + (a_n - \beta_n P)x_n = 0 \tag{4}$$

We thus want to maximize P such that (4) is satisfied.

We will make a modification of our objective function, in that we try to maximize[1]

$$f = \sum_{j=1}^{n} (a_j - \beta_j \bar{P})x_j$$

Our problem is transformed to a general problem of parametric linear programming; it can now be solved using the simplex-method.

Our initial basis consists of the column vectors corresponding to the slack-variables. Our initial simplex-tableau is shown in Table 1, where z_j is defined as

$$z_j = \sum_{i=1}^{m} c_i^B \bar{a}_{ij},$$

$\bar{a}_{ij}$ being the current element in row i and column j of the simplex-tableau.

c^B | c_j	$a_1-\beta_1\bar{P}$	$a_2-\beta_2\bar{P}$	....	$a_n-\beta_n\bar{P}$	0	0	....	0	
	x_1	x_2	....	x_n	x_{n+1}	x_{n+2}	....	x_{n+m}	b
0	a_{11}	a_{12}	....	a_{1n}	1	0	....	0	b_1
0	a_{21}	a_{22}	....	a_{2n}	0	1	....	0	b_2
—	....	....	....	....	....	....	....	....	—
—	....	....	....	....	....	....	....	....	—
—	....	....	....	....	....	....	....	....	—
0	a_{m1}	a_{m2}	....	a_{mn}	0	0	....	1	b_m
$z_j - c_j$	$\beta_1\bar{P}-a_1$	$\beta_2\bar{P}-a_2$	....	$\beta_n\bar{P}-a_n$	0	0	....	0	

TABLE 1.

[1] Proof in Dinkelbach [3].

Let $M=\{j_1, \ldots, j_m\}$ be the set of indices for the variables in the basis before each iteration. A feasible solution is one that satisfies (3). The vector $x=(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})=(0, \ldots, 0, b_1, \ldots, b_m)$ is thus a feasible solution. For it to be an optimal solution we must have $z_j-c_j\geq 0$ for all j. This implies $\beta_j\bar{P}-a_j\geq 0$ for all $j \Rightarrow \bar{P}\geq\frac{a_j}{\beta_j}$ for all j. Let K_1 be $\max\left\{\frac{a_j}{\beta_j}\right\}$. The condition for $x=(0, \ldots, 0, b_1, \ldots, b_m)$ to be an optimal basic solution is that $\bar{P}\geq K_1$. We calculate $P=\frac{a_0}{\beta_0}$. If $P\geq K_1$, we have found the solution x which makes P maximal. If $P<K_1$, the solution is not optimal.

Assume that the solution is not optimal. By setting $\bar{P}=K_1$, we will have at least one j not in M such that $z_j-c_j=0$. Suppose this index is k. We then choose x_k as the new basic variable.

The variable to be removed from the basis is found in the usual way, and a new simplex-tableau is computed. This gives us a new set of $z_j-c_j=\bar{\beta}_j\bar{P}-\bar{a}_j$ and a new vector x which is an optimal basic solution if $z_j-c_j\geq 0$ for all j. This condition will be satisfied if $K_2\leq\bar{P}\leq K_1$, where K_2 is a new constant computed by $K_2=\max\left\{\frac{\bar{a}_j}{\bar{\beta}_j}\right\}$. We then compute

$$P=\frac{a_0+a_kx_k}{\beta_0+\beta_kx_k}$$

If $K_2\leq P\leq K_1$, P is maximal. If not, i.e. if $P<K_2$, we set $\bar{P}=K_2$ and find the new variable to enter into basis.

This procedure is continued until P actually lies within the required interval which makes all $z_j-c_j\geq 0$.

To illustrate the method, we will go through a numerical example. We have the following problem.

$$\operatorname{Max} P=\frac{-20+5x_1+3x_2+2x_3}{4+2x_1+3x_2+x_3}$$

subject to

$$\begin{aligned}
x_1 \qquad\qquad &\leq 10\\
x_2 \qquad &\leq 8\\
x_3 &\leq 6\\
x_1+x_2+x_3 &\leq 18\\
x_1 \qquad +2x_3 &\leq 18\\
x_i\geq 0,\ i&=1,2,3
\end{aligned}$$

The objective function can be transformed to: Max$P$ such that $(-20-4P)+(5-2P)x_1+(3-3P)x_2+(2-P)x_3=0$. After introduction of slack-variables $x_4, \ldots, x_8$, we can form the initial simplex-tableau (see Table 2), where our objective function to be maximized is

$$f=(5-2\bar{P})x_1+(3-3\bar{P})x_2+(2-\bar{P})x_3 \, .$$

c^B \ c_j	$5-2\bar{P}$	$3-3\bar{P}$	$2-\bar{P}$	0	0	0	0	0	
	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
0	1	0	0	1	0	0	0	0	10
0	0	1	0	0	1	0	0	0	8
0	0	0	1	0	0	1	0	0	6
0	1	1	1	0	0	0	1	0	18
0	1	0	2	0	0	0	0	1	18
z_j-c_j	$2\bar{P}-5$	$3\bar{P}-3$	$\bar{P}-2$	0	0	0	0	0	

Table 2.

We have $x_1=0$, $x_2=0$, $x_3=0$. We want to investigate whether this solution makes P maximal or not. The condition $z_j-c_j \geq 0$ gives:

$$\left.\begin{aligned} 2\bar{P}-5\geq 0 &\Rightarrow \bar{P}\geq 2{,}5 \\ 3\bar{P}-3\geq 0 &\Rightarrow \bar{P}\geq 1 \\ \bar{P}-2\geq 0 &\Rightarrow \bar{P}\geq 2 \end{aligned}\right\} \Rightarrow \bar{P}\geq 2{,}5$$

$P=\dfrac{-20}{4}=-5$, which means that the solution is not optimal.

If we put $\bar{P}=2{,}5$, we have $z_1-c_1=0$, and x_1 is our new basic variable.

Pivoting is now performed, and the new simplex-tableau is shown in Table 3.

c^B \ c_j	$5-2\bar{P}$	$3-3\bar{P}$	$2-\bar{P}$	0	0	0	0	0	
	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
$5-2\bar{P}$	1	0	0	1	0	0	0	0	10
0	0	1	0	0	1	0	0	0	8
0	0	0	1	0	0	1	0	0	6
0	0	1	1	-1	0	0	1	0	8
0	0	0	2	-1	0	0	0	1	8
z_j-c_j	0	$3\bar{P}-3$	$\bar{P}-3$	$5-2\bar{P}$	0	0	0	0	

Table 3.

Our solution is $x_1=10$, $x_2=0$, $x_3=0$. The condition $z_j-c_j\geq 0$ implies:

$$\left.\begin{array}{l} 3\bar{P}-\ \ 3\geq 0\Rightarrow \bar{P}\geq 1 \\ \bar{P}-\ \ 2\geq 0\Rightarrow \bar{P}\geq 2 \\ 5-2\bar{P}\geq 0\Rightarrow \bar{P}\leq 2,5 \end{array}\right\} \Rightarrow 2\leq \bar{P}\leq 2,5$$

$$P=\frac{-20+5\cdot 10}{4+2\cdot 10}\cdot=1,25$$

and since $P<2$, this P is not maximal either.

x_3 is the new basic variable, and the resulting simplex-tableau is shown in Table 4.

c^B \ c_j	$5-2\bar{P}$	$3-3\bar{P}$	$2-\bar{P}$	0	0	0	0	0	
	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
$5-2\bar{P}$	1	0	0	1	0	0	0	0	10
0	0	1	0	0	1	0	0	0	8
0	0	0	0	$\frac{1}{2}$	0	1	0	$-\frac{1}{2}$	2
0	0	1	0	$-\frac{1}{2}$	0	0	1	$-\frac{1}{2}$	4
$2-\bar{P}$	0	0	1	$-\frac{1}{2}$	0	0	0	$\frac{1}{2}$	4
z_j-c_j	0	$3\bar{P}-3$	0	$4-1,5\bar{P}$	0	0	0	$1-0,5\bar{P}$	

Table 4.

Here we have $x_1=10$, $x_2=0$, $x_3=4$. The condition $z_j-c_j\geq 0$ now gives:

$$\left.\begin{array}{l} 3\bar{P}-\ \ \ \ 3\geq 0\Rightarrow \bar{P}\geq 1 \\ 4\ \ -1,5\bar{P}\geq 0\Rightarrow \bar{P}\leq 2,67 \\ 1\ \ -0,5\bar{P}\geq 0\Rightarrow \bar{P}\leq 2 \end{array}\right\} \Rightarrow 1\leq \bar{P}\leq 2$$

$$P=\frac{-20+50+8}{4+20+4}\cdot=\frac{38}{28}=1,3571\ ,$$

i.e. $1\leq P\leq 2$, which means that P has reached its maximum.

We see that if the constant part of the numerator had been lower than -30, we would have had $P<1$, and it would then also have been optimal to introduce x_2 into the basis.

2) *Dual evaluators*

In the optimal simplex-tableau for a linear programming problem one will find the dual evaluator for constraint no. i as the value of $z_{n+i}-c_{n+i}$, i.e. in the column corresponding to the slack-variable for constraint no. i. The dual evaluator for a constraint can only be positive if the corresponding slack-variable is equal to 0, i.e. if the inequality no. i is tight; the dual evaluator no. i tells how much the objective function would increase per unit increase in the right-hand side b_i under the assumption that the increase does not require change of basis in the optimal solution.

While each $z_j - c_j$ in the linear case consists of only one number, in the initial tableau $-c_j$, the $z_j - c_j$ for our problem will generally contain two components, in the initial tableau $-\alpha_j + \bar{\beta}_j \bar{P}$. During the iterations α_j and β_j are changed in exactly the same way as they would have been if the same basis had been reached with the numerator or denominator alone as the objective function. It can be shown that the $\bar{\alpha}_j$'s and $\bar{\beta}_j$'s for our optimal tableau serve as dual evaluators for the numerator and denominator respectively given our optimal basis; our optimal basis is generally not optimal relative to the numerator or denominator alone as objective function.

If we write for the optimal tableau: $z_{n+i} - c_{n+i} = \bar{\alpha}_i - \bar{\beta}_i \bar{P}$, $\bar{\alpha}_i$ thus indicates how much the numerator would increase per unit increase in b_i, while $\bar{\beta}_i$ tells how much the denominator would change, assuming as before, no change in basis. If we denote the numerator of our optimal solution N and the denominator D, we can find how much the objective function P would increase with an increase of b_i with k_i units, $i = 1, \ldots, m$, by computing

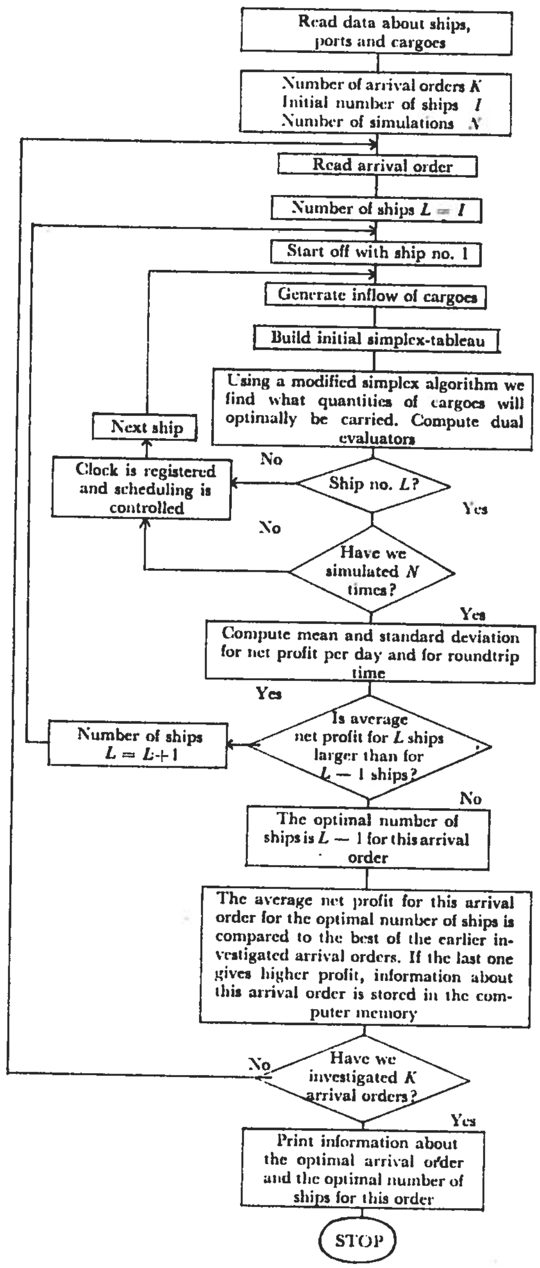
$$\frac{N + \sum_{i=1}^{m} k_i \bar{\alpha}_i}{D + \sum_{i=1}^{m} k_i \bar{\beta}_i} - \frac{N}{D}.$$

Going back to our numerical example, Table 4, we see that two dual evaluators are different from 0, namely those corresponding to the constraints no. 1 and 5. The first one tells that an increase of b_1 by one unit from 10 to 11 would result in an increase of P by

$$\frac{38+4}{28+1{,}5} - \frac{38}{28} = 0{,}0666.$$

APPENDIX B

Flow-chart for the main parts of the simulation model.



If b_5 increased simultaneously from 18 to 19 units, P would instead increase by

$$\frac{38+4+1}{28+1{,}5+0{,}5}-\frac{38}{28}=0{,}0762$$

Because of the non-linear objective function, the dual variables for our problem only evaluates infinitesimal, i.e. infinitely small increases in the right-hand side; it can be shown[1], however, that these dual variables can be calculated by the following formula:

$$u_i=\frac{1}{D}\bar{a}_i-\frac{N}{D^2}\bar{\beta}_i=\frac{1}{D}(\bar{a}_i-P\bar{\beta}_i)$$

For the first constraint we have:

$$u_1=\frac{1}{28}(4-1{,}3571\cdot 1{,}5)=0{,}0702$$

We see that the marginal value per unit is a little higher than the total value for the first unit.

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[1] See Kydland [5].