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NONCOOPERATIVE AND DOMINANT PLAYER SOLUTIONS IN DISCRETE DYNAMIC GAMES*

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1. INTRODUCTION

ECONOMISTS' INTEREST IN THE PROBLEM of how to apply economic instruments so as to reach certain targets dates at least back to the work of Meade [**17**] and Tinbergen [**31**]. Their approach, however, differed from the optimal control formulations in terms of linear systems with quadratic objective function later discussed by Simon [**26**], Theil [**29**], and Holt *et al.* [**12**]. Early applications of control theory were made by Bogaard and Theil [**1**], Holt [**11**] and Theil [**30**]. In the last couple of years we have seen a continuance, if not revival, of interest in control type macroeconomic planning models. Some economists have done a good job in making results from the control literature available in a form convenient to the economist, stressing the aspects of control theory likely to be most useful to him. They have also done theoretical work on problems of interest in economic models, such as problems arising from the stochastic nature of the models, including stochastic coefficients and learning over time.[2]

In all the papers mentioned above it was assumed that there is only one decision maker, or at least that there is only one objective function. However, in actual life the instruments of the economy can be under the control of different policymakers who each may have conflicting views on target values or the relative importance of the targets. In the U.S., for instance, it is unlikely that the Congress or the Administration, controlling fiscal policy, and the Federal Reserve Board, controlling monetary policy, hold the same views on what the targets of their policies should be. It is not clear either that much cooperation is taking place between them. James L. Pierce on the Board of Governors of the Federal Reserve System described the situation as follows:[3] "One of our biggest problems is predicting fiscal policy over the policy horizon. As you know, in the United States, monetary and fiscal policy are determined separately."

What particularly complicates the situation, is that one cannot just predict the policy of the other policymaker and go ahead taking that as a given. One must also, if one is rational, take into account the effect that one's own policy

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[1] This paper is based on a chapter of my Ph. D. dissertation submitted to Carnegie-Mellon University. I am indebted to David Cass, Robert S. Kaplan, and Edward C. Prescott for their advice and encouragement.

[2] Some recent papers are Chow [**3, 4, 5, 6**], Holbrook [**10**], Pindyck [**21**], and Prescott [**22, 23**]. See also the papers presented at the NBER-NSF Conference on Stochastic Control and Economic Systems in Chicago, June 7–9, 1973. Some of these papers are published in Annals of Economic and Social Measurement, III (January, 1974). Zellner [**32**, (Chapter XI)] has a good discussion of the stochastic aspects of optimal control.

[3] Pierce [**20**, (12)].

will have on the other policymaker's policies in the future. This discussion suggests a formulation of decentralized policymaking as a dynamic game.

In this paper we present such a game-theoretic framework. The decision environment is described by a set of linear difference equations with additive disturbances. Each player is assumed to behave as if he minimizes the expected value of a preference (loss) function which is approximated by a quadratic function.[4] The objective functions are in general different for each player.

In making his decision each player has certain expectations as to what the other players will do. In this paper we study equilibrium solutions, and leave problems of stability aside.[5] We assume that no coalitions are formed, and we shall first study noncooperative solutions (also called Nash equilibria). The equilibrium solution is such that, given the decisions of the other players, no player has any incentive to change his decision rule (or regret having chosen it). We might say that the players are assumed to have rational expectations in the sense that the expectation of the others' actions turn out to be the actual outcome.[6] However, in dynamic games there turn out to be more than one possible solution concept which give different solutions, even in the absence of uncertainty. A central topic of this paper is therefore the evaluation of open loop and feedback solutions as possible candidates for an equilibrium solution.

In Section 2 we present the noncooperative feedback solution to an n-person dynamic game. We also comment on how the open loop solution could be computed. In Section 3 open loop and feedback solutions are compared, and we try to give an understanding of why they are different.

Casual observation of macroeconomic policymaking in the U.S. indicates that the assumption of noncooperative solution is not necessarily the one that describes reality best. It seems that the Congress with certain intervals will announce its tax and spending policy. Given that, the Federal Reserve Board will try to meet its objectives by adjusting its monetary instruments. We might say that the Congress is a dominant player who announces his decision first,

[4] Admittedly our assumptions are such that uncertainty will play only a minor role. We shall be able to use the well known certainty equivalence property (cf. Simon [**26**] or Theil [**29**]) and compute the relevant decision rules from a problem where the stochastic variables are replaced by their mean values. However, the implications of uncertainty will be pointed out wherever necessary, and even when included in such a simple manner, the presence uncertainty may still lead to key results, as can be seen for instance in [**15**].

[5] By stability we refer to the property that under reasonable assumptions the decision rules will move towards the equilibrium decision rules if the system is subject to shocks, or if the players initially have incomplete information about how the system works. By contrast, the literature on the assignment problem, which also deals with decentralized policymaking, although not from an optimization point of view, is mainly concerned with the stability problem. The so-called principle of effective market classification states that each policy instrument should be directed towards that target on which it has relatively the greatest impact. Mundell, who first posed the problem [**18**], has stated that the principle of effective market classification is basically a mathematical proposition which implies that, instead of letting each institution deal with several problems (or goals), responsibilities should be allocated to various authorities in such a way as to ensure stability of the system (see [**7**, (129)]).

[6] See Muth [**19**] or Lucas and Prescott [**16**] for discussions of the concept of rational expectations.

while the other player (the Fed) decides what his optimal decision is, taking the decision of the dominant player as given. In making his decision, the dominant player takes into account the reaction functions of the nondominant players. In equilibrium he correctly foresees these functions. In the case where there is more than one nondominant player, these are assumed to behave noncooperatively among themselves. The feedback solution for this case is given in Section 4. We point out that the open loop solution is likely to fail as an equilibrium concept.

The dominant player solution is a special case of a much more general solution concept into which almost any kind of hierarchical structure can be incorporated. We indicate how our results can be generalized in that direction.

A few short remarks on the infinite horizon problem are made in Section 5. The final section offers some concluding comments, in particular with regard to economic applications.

2. NONCOOPERATIVE SOLUTIONS IN DISCRETE DYNAMIC GAMES[7]

To simplify notation we shall assume, with little loss of generality, that each of the n players has control over one instrument only. More general cases can be taken care of in our model by specifying the same preference function for more than one player.

In the following x_t will be an n-dimensional vector, y_t is m-dimensional, and the remaining dimensions will be obvious. Define

$$w_i(y_t) = p_i'y_t + \frac{1}{2}y_t'Q_iy_t\,.$$

Then the optimization problem for player i is

$$\underset{\{x_{i1},\ldots,x_{iT}\}}{\text{minimize}}\; E\left\{\sum_{t=1}^{T}\beta_i^{t-1}w_i(y_t)\right\}$$

subject to[8]

$$y_t = Ay_{t-1} + Bx_t + c + \varepsilon_t$$

y_0 given

$0 < \beta_i < 1$

ε_t identically and independently distributed over time with mean 0 and finite covariance matrix Σ_ε.

Given decisions of the other players.

[7] From the viewpoint of game theory this section does not really offer any new results. Games in discrete time are, of course, quite similar to games in continuous time. Papers on differential games discussing both open loop and closed loop controls for linear-quadratic systems are Case [2], Foley and Schmitendorf [8], Ho [9], and Starr and Ho [27, 28]. The dominant player problem, on the other hand, has only recently received a little attention in the game literature, and the two interesting papers by Simaan and Cruz [24, 25] should be mentioned.

[8] The results of this paper for the finite horizon case can easily be extended to include exogenous variables.

It is well known that linear systems with any finite number of lags can be written on the above form after a suitable redefinition of variables (see e.g., Chow **[5, 6]** or Prescott **[22]**). Also, dependence of the objective function on the instruments can be incorporated. For example, assume that it is considered costly to change the instruments fast. Then we might have

$$w(x_{t-1}, x_t, y_t) = p'y_t + \frac{1}{2} y_t'Qy_t + \frac{1}{2}(x_t - x_{t-1})'K(x_t - x_{t-1})$$

subject to

$$y_t = A_1 y_{t-1} + A_2 y_{t-2} + B_1 x_t + B_2 x_{t-1} + c\,.$$

This we can write as

$$w(x_{t-1}, x_t, y_t) = [p \;\; 0 \;\; 0 \;\; 0]\; [y_t \;\; y_{t-1} \;\; x_t \;\; x_{t-1}]' + \frac{1}{2} \begin{bmatrix} y_t \\ y_{t-1} \\ x_t \\ x_{t-1} \end{bmatrix}' \begin{bmatrix} Q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & K & -K \\ 0 & 0 & -K & K \end{bmatrix} \begin{bmatrix} y_t \\ y_{t-1} \\ x_t \\ x_{t-1} \end{bmatrix}$$

$$\begin{bmatrix} y_t \\ y_{t-1} \\ x_t \\ x_{t-1} \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & B_2 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ x_{t-1} \\ x_{t-2} \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \\ I \\ 0 \end{bmatrix} x_t + \begin{bmatrix} c \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which by an obvious redefinition of variables can be written on the form we are using.

The assumptions on ε_t clearly allow us to use the certainty equivalence property and ignore the disturbances in computing the first-period decision rules. We thus omit ε_t except where it makes a difference.

Backward induction will now be used to compute the equilibrium feedback solution. We let $v_{it}(y_{t-1})$ denote the total loss incurred to player i in following an optimal equilibrium policy from period t to the end of the horizon when the state variables initially are y_{t-1}. We define $v_{T+1} \equiv 0$.

To make clear the distinction between the decision variable under the player's own control, and expected values of others' decisions, we shall denote the former by z_i for player i, and the latter by $x_j, j \neq i$. Then the optimization problem for player i in period t can be formulated as follows, where b_i is column i of B

$$v_{it}(y_{t-1}) = \min_{z_{it}} \{w_i(y_t) + \beta_i v_{i,t+1}(y_t)\}$$

subject to

$$y_t = Ay_{t-1} + \sum_{\substack{j=1 \\ j \neq i}}^{n} b_j x_{jt} + b_i z_{it} + c\,. \tag{2.1}$$

DEFINITION. An equilibrium decision at time t is a decision x_t^0 such that

$$\min_{z_{it}} \Big\{ w_i \Big(Ay_{t-1} + \sum_{\substack{j=1 \\ j \neq i}}^{n} b_j x_{jt}^0 + b_i z_{it} + c \Big)$$
$$+ \beta_i v_{i,t+1} \Big(Ay_{t-1} + \sum_{\substack{j=1 \\ j \neq i}}^{n} b_j x_{jt}^0 + b_i z_{it} + c \Big) \Big\}$$
$$= w_i(Ay_{t-1} + Bx_t^0 + c) + \beta_i v_{i,t+1}(Ay_{t-1} + Bx_t^0 + c) , \qquad i = 1, \ldots, n.$$

This definition implies that, in order to get the equilibrium solution, we shall be looking for a solution such that $x = z$. This means that for no player does it pay to change his decision for any time period.

We define the following notation

$$H_t = \begin{bmatrix} b_1'(Q_1 + \beta_1 S_{1,t+1}) \\ \vdots \\ b_n'(Q_n + \beta_n S_{n,t+1}) \end{bmatrix} \quad \text{and} \quad k_t = \begin{bmatrix} b_1'(p_1 + \beta_1 r_{1,t+1}) \\ \vdots \\ b_n'(p_n + \beta_n r_{n,t+1}) \end{bmatrix}$$

THEOREM 1. *Assume that*

(i) $b_i'(Q_i + \beta_i S_{i,t+1})\, b_i > 0$ *for* $i = 1, \ldots, n$, *and* $t = 1, \ldots, T$.

(ii) $|H_t B| \neq 0$ *for* $t = 1, \ldots, T$.

Then we can compute recursively the unique equilibrium solution for period t

$$z_t^0 = x_t^0 = -(H_t B)^{-1}(k_t + H_t c + H_t A y_{t-1})$$
$$\equiv d_t + E_t y_{t-1} ,$$

and the value function for player i, $i = 1, \ldots, n$, *which is of the form*

$$v_{it}(y_{t-1}) = \nu_{it} + r_{it}' y_{t-1} + \frac{1}{2} y_{t-1}' S_{it} y_{t-1} ,$$

where $v_{i,T+1} \equiv 0$, *and where* ν_{it}, r_{it} *and* S_{it} *are determined by*[9]

$$S_{it} = (A + BE_t)'(Q_i + \beta_i S_{i,t+1})(A + BE_t)$$
$$r_{it} = (A + BE_t)'[p_i + \beta_i r_{i,t+1} + (Q_i + \beta_i S_{i,t+1})(Bd_t + c)]$$
$$\nu_{it} = \beta_i \nu_{i,t+1} + \Big[p_i + \beta_i r_{i,t+1} + \frac{1}{2}(Q_i + \beta_i S_{i,t+1})(Bd_t + c)\Big]'(Bd_t + c) .$$

PROOF. Assume that $v_{i,t+1}$ has been found by backward induction for all i. It is trivial to show that each $v_{i,t+1}$ is quadratic, say,

$$v_{i,t+1}(y_t) = \nu_{i,t+1} + r_{i,t+1}' y_t + \frac{1}{2} y_t' S_{i,t+1} y_t , \qquad i = 1, \ldots, n.$$

[9] With a stochastic ε_t we have to add to ν_{it} the term 1/2 trace $((Q_i + \beta_i S_{i,t+1})\Sigma_\varepsilon)$. Note that if the state vector has been augmented as describedo n page 324, a lot of computations are saved by remembering that the corresponding added parts of r, S, d, and E are zero. For many purposes we are only interested in E, which characterizes the solution. In that case we need not compute the recursive relations for d, r, and ν.

We can now write

$$v_{it}(y_{t-1}) = \min_{z_{it}} \left\{\beta_i v_{i,t+1} + (p_i + \beta_i r_{i,t+1})' y_t + \frac{1}{2} y_t'(Q_i + \beta_i S_{i,t+1}) y_t\right\}. \tag{2.2}$$

Substituting for y_t from (2.1) and differentiating we get the first-order conditions, which by assumption (i) are both necessary and sufficient for a minimum:

$$b_i'(p_i + \beta_i r_{i,t+1}) + b_i'(Q_i + \beta_i S_{i,t+1}) \times \left(Ay_{t-1} + \sum_{\substack{j=1 \\ j \neq i}}^{n} b_j x_{jt} + b_i z_{it} + c\right) = 0\,, \qquad i = 1, \ldots, n. \tag{2.3}$$

The decision function for player i is

$$z_{it} = -\frac{1}{b_i'(Q_i + \beta_i S_{i,t+1}) b_i} \Big[b_i'(p_i + \beta_i r_{i,t+1} + (Q_i + \beta_i S_{i,t+1})c) + b_i'(Q_i + \beta_i S_{i,t+1}) A y_{t-1} + \sum_{\substack{j=1 \\ j \neq i}}^{n} b_i'(Q_i + \beta_i S_{i,t+1}) b_j x_{jt} \Big],$$

and in equilibrium we have, using assumption (ii)

$$z_t^0 = x_t^0 = -(H_t B)^{-1}(k_t + H_t c + H_t A y_{t-1}) \equiv d_t + E_t y_{t-1} \tag{2.4}$$

which is unique.

We can now substitute from (2.1) and (2.4) into (2.2), giving both the left-hand side and the right hand side in terms of y_{t-1}. Comparing the coefficients for the second-degree term, the first-degree term, and the constant, respectively, we get the recursive relations for S_{it}, r_{it}, and ν_{it}. This completes the proof.

Assumption (i) is relatively weak in that it can be satisfied for player i even if some diagonal elements of Q_i are negative; that is, Q_i need not even be positive semidefinite as is usually required for the standard control problem.

The implications of Assumption (ii) may be less obvious. Hopefully Figure 1 is helpful in this regard. Assume for simplicity that there are only two players, with control over x_{1t} and x_{2t}, respectively. Also define the following notation

$$u_{it}(x_t, y_{t-1}) = w_i(y_t) + \beta_i v_{i,t+1}(y_t)\,.$$

For a given y_{t-1}, then, the possible decisions at time t imply values for y_t, which again imply values of u_{1t} and u_{2t}. This means that, for a given time period t and a given value of y_{t-1}, we can draw constant u_{it}-contours in a diagram with z_{1t} (or x_{1t}) and z_{2t} (or x_{2t}) along the axes. We know that $u_{it}(x_t | y_{t-1})$ is a quadratic function. Given what player 1, say, thinks that player 2 will do as represented by x_{2t} he will choose z_{1t} according to the following equation, where α_1 is some constant

$$b_1'(Q_1 + \beta_1 S_{1,t+1}) b_1 z_{1t} + b_1'(Q_1 + \beta_1 S_{1,t+1}) b_2 x_{2t} + \alpha_1 = 0\,.$$

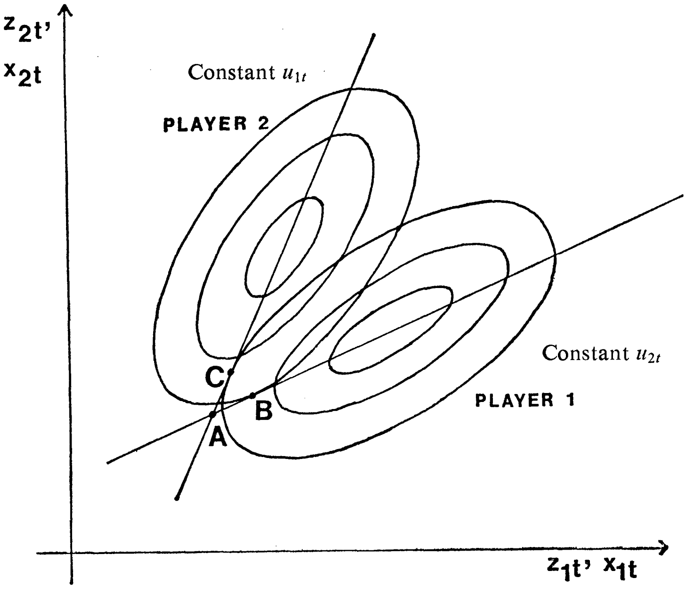

FIGURE 1

This is a straight line, and is shown in Figure 1 going through the contours drawn for player 1.

Similarly, player 2 will behave according to

$$b_2'(Q_2 + \beta_2 S_{2,t+1})b_1 x_{1t} + b_2'(Q_2 + \beta_2 S_{2,t+1})b_2 z_{2t} + \alpha_2 = 0\,.$$

This is the straight line through the contours for player 2. The equilibrium solution is the point A where the two reaction lines cross. Assumption (ii) guarantees that the two lines do not have the same slope, in which case there would either be infinitely many solutions if the lines were identical, or no solution at all if they were not identical.

Before ending this section we shall try to give an understanding, without going into details, of how the open loop solution could be obtained. The most instructive way to set up the problem is probably to write out all the equations for all the T time periods at once and solve. The problem for player i would then look as follows[10]

$$\underset{\{x_{i1},\ldots,x_{iT}\}}{\text{minimize}} \begin{bmatrix} p_i \\ \beta_i p_i \\ \vdots \\ \beta_i^{T-1} p_i \end{bmatrix}' \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{bmatrix} + \frac{1}{2} \begin{bmatrix} y_1 \\ \vdots \\ \vdots \\ y_T \end{bmatrix}' \begin{bmatrix} Q_i & 0 & \cdots & 0 \\ 0 & \beta_i Q_i & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \beta_i^{T-1} Q_i \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ \vdots \\ y_T \end{bmatrix}$$

subject to

[10] For simplicity we do not distinguish between x_i and z_i as we did above.

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{bmatrix} = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^T \end{bmatrix} y_0 + \begin{bmatrix} B & 0 & \cdots & 0 \\ AB & B & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ A^{T-1}B & A^{T-2}B & \cdots & B \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_T \end{bmatrix} + \begin{bmatrix} I \\ I + A \\ \vdots \\ \sum_{i=0}^{T-1} A^i \end{bmatrix} c$$

y_0 given.

Differentiating with respect to x_{it}, $t = 1, \ldots, T$, player i obtains the first-order conditions for a minimum in the form of T equations. The equilibrium open loop solution is obtained by solving the system of nT equations from all of the players, giving

$$\begin{bmatrix} x_1 \\ \vdots \\ x_T \end{bmatrix} = \begin{bmatrix} d_1^* \\ \vdots \\ d_T^* \end{bmatrix} + \begin{bmatrix} E_1^* \\ \vdots \\ E_T^* \end{bmatrix} y_0 ,$$

where the starred d's and E's depend on the parameters of the problems for the n players. We now proceed to compare the two types of solutions, in particular with regard to their suitability as an equilibrium concept.

3. A COMPARISON OF OPEN LOOP AND FEEDBACK SOLUTIONS

The difference between the two solution concepts will become clearer if we use a somewhat more general and compact notation. Since the argument does not depend on the presence of random disturbances, we consider for simplicity the deterministic version of the model.

The open loop solution is found by solving

$$\underset{z_{i1}, \ldots, z_{iT}}{\text{minimize}} \sum_{t=1}^{T} \beta_i^{t-1} w_i(y_t^i)$$

subject to

$$y_t^i = f_t^i(y_0, x_1^i, \ldots, x_t^i, z_{i1}, \ldots, z_{it}) , \qquad t = 1, \ldots, T$$

$$y_0, x_1^i, \ldots, x_T^i \text{ given,}$$

where

$$x_t^i = (x_{1t}, \ldots, x_{i-1,t}, x_{i+1,t}, \ldots, x_{nt})$$

is player i's expectation of the other players' decisions.

The solutions to this problem for each time period are n mappings

$$y_0, x_1^i, \ldots, x_T^i \to z_{it} , \qquad i=1, \ldots, n;\ t = 1, \ldots, T$$

derived from the first-order conditions for a minimum. The assumption of non-cooperative solution implies that[11]

$$z_t = x_t = g_t^*(y_0) , \qquad t = 1, \ldots, T.$$

[11] In the stochastic case we would have $z_t = x_t = g_t^*(y_0, \varepsilon_1, \ldots, \varepsilon_{t-1})$, $t = 1, \ldots, T$.

For the feedback solution we assume that at each time period a noncooperative solution is chosen as a function of the state variable at that time. This means that each player, instead of taking as given a sequence of decisions for the other players, takes as given decision rules for each time period t that are functions of the state variables y_{t-1}. The optimization problem for player i, then, is

$$\underset{z_{i1},\ldots,z_{iT}}{\text{minimize}} \sum_{t=1}^{T} \beta_i^{t-1} w_i(y_t^i)$$

subject to

$$y_t^i = f^i(y_{t-1}, x_t^i, z_{it}) ,$$
$$y_0, x_t^i = g_t^i(y_{t-1}) \text{ given.}$$

Let $v_{i,t+1}(y_t)$, $i = 1, \ldots, n$, be the total values of the succession of noncooperative solutions

$$z_s = x_s = g_s(y_{s-1}) , \qquad s = t+1, \ldots, T.$$

Then we have

$$v_{it}(y_{t-1}, x_t^i) = \min_{z_{it}} \{w_i(y_t^i) + \beta_i v_{i,t+1}(y_t^i)\}$$

subject to

$$y_t^i = f^i(y_{t-1}, x_t^i, z_{it})$$
$$y_{t-1}, x_t^i \text{ given.}$$

The solutions are mappings

$$y_{t-1}, x_t^i \to z_{it} , \qquad i = 1, \ldots, n,$$

which by the assumption of noncooperative solution imply

$$z_t = x_t = g_t(y_{t-1}) .$$

Given this solution, we can now evaluate the value functions as functions of y_{t-1} only, and $v_{it}(y_{t-1})$, $i = 1, \ldots, n$, thus gives the total value of the succession of noncooperative solutions from period t until the end of the horizon.

It is clear that the feedback solution is different from the open loop solution, even ignoring the randomness.[12] Intuitively, the reason for this is as follows.

[12] A simple numerical example may be useful. Assume that the problems of two noncooperative players are

$$\underset{x_{i1},x_{i2}}{\text{maximize}} \sum_{t=1}^{2} \left[(1 - y_{1t} - y_{2t})y_{it} - \frac{1}{2} x_{it}^2\right]$$

subject to

$$y_{it} = y_{i,t-1} + x_{it} , \qquad i = 1, 2,$$
$$y_0 \text{ given.}$$

The open loop decision rules for player 1 are (since the two problems are symmetric, the decision rules are symmetric)

$$x_{11} = -.6947y_{10} - .0947y_{20} + .2632$$
$$x_{12} = -.1789y_{10} + .0211y_{20} + .0526$$

while the corresponding feedback solutions are

(*Continued on next page*)

In making his decision, player i knows that it will affect the state variables. A change in the state variables will change the other players' decisions in the future. This change in the other players' decisions will have an effect on future losses for player i. This fact is taken into account in the feedback solution when player i makes his decision.

Both approaches may be said to represent noncooperative solutions in some sense. However, the feature described above seems to lend more realism to the feedback solution as an equilibrium concept. It seems reasonable in many models to assume that each decision maker will determine the effect of his decision on the state variables and consider how other players will react in the future. For instance, in an oligopolistic industry each firm may take into consideration the effect of its decision on market shares and assume that the other players will react in certain ways to different sizes of the market shares. This seems particularly reasonable if we think in terms of stability, that is, view the equilibrium solution as the end result of a process with all the players groping for decision rules that are such that, given the others' actions, nobody has any incentive to change the rule, and where forecasting errors are corrected as the players learn more about the other players' behavior.

4. DOMINANT PLAYER SOLUTIONS

In this section we assume that one player, say player n, is dominant in the sense that he may announce his decision first, thereby taking into account the effect of his decision upon the other players' decisions. When he has announced his decision, the other players behave noncooperatively with the decision of the dominant player as given.

We shall outline the computations of the feedback solution. The open loop solution will then be obvious. As in Section 2 we assume that the problems for the n players are such that the first-order conditions will determine both necessary and sufficient conditions for a minimum, and that the resulting decision functions are unique.

The problems of the $n - 1$ nondominant players are as described in Section 2. There are $n - 1$ first-order conditions as in (2.3), and given the decision of player n, we get a unique noncooperative solution at time t of the form

$$x_{it}^{0} = \gamma_{it} + g_{it}y_{t-1} + \eta_{it}x_{nt}, \qquad i = 1, \ldots, n-1. \tag{4.1}$$

The optimization problem for the dominant player in period t is

$$v_{nt}(y_{t-1}) = \min_{x_{nt}} \{w_n(y_t) + \beta_n v_{n,t+1}(y_t)\} \tag{4.2}$$

(*Continued*)

$$x_{11} = -.6889y_{10} - .0992y_{20} + .2715$$

$$x_{12} = -.6250y_{11} - .1250y_{21} + .2500.$$

If the initial state variables were $y_{10} = y_{20} = .1$, then the open loop solution for player 1 would be $x_{11} = .1843$ and $x_{12} = .0368$, while the feedback solution would be $x_{11} = .1927$ and $x_{12} = .0305$.

subject to

$$y_t = Ay_{t-1} + Bx_t + c$$
$$x_{it} = \gamma_{it} + g_{it}y_{t-1} + \eta_{it}x_{nt}, \qquad i = 1, \dots, n-1.$$

The last two expressions can be combined as follows

$$\begin{aligned} y_t &= \Big(A + \sum_{j=1}^{n-1} b_j g_{jt}\Big) y_{t-1} + \Big(b_n + \sum_{j=1}^{n-1} b_j \eta_{jt}\Big) x_{nt} + \Big(c + \sum_{j=1}^{n-1} b_j \gamma_{jt}\Big) \\ &\equiv A_t^+ y_{t-1} + b_{nt}^+ x_{nt} + c_t^+ . \end{aligned} \tag{4.3}$$

Since we know that $v_{n,t+1}(y_t)$ is of the form

$$v_{n,t+1}(y_t) = \nu_{n,t+1} + r'_{n,t+1} y_t + \frac{1}{2} y'_t S_{n,t+1} y_t ,$$

we can write (4.2) as

$$\begin{aligned} v_{nt}(y_{t-1}) = \min_{x_{nt}} \Big\{ &\beta_n \nu_{n,t+1} + (p_n + \beta_n r_{n,t+1})' y_t \\ &+ \frac{1}{2} y'_t (Q_n + \beta_n S_{n,t+1}) y_t \Big\} . \end{aligned}$$

Differentiating, and remembering (4.3), we get

$$b_{nt}^{+\prime}(p_n + \beta_n r_{n,t+1}) + b_{nt}^{+\prime}(Q_n + \beta_n S_{n,t+1})(A_t^+ y_{t-1} + b_{nt}^+ x_{nt} + c_t^+) = 0$$

which can be solved for x_{nt} to give an equation of the form

$$x_{nt}^0 = \delta_{nt} + e_{nt} y_{t-1} . \tag{4.4}$$

Substituting (4.4) into (4.1) we get

$$\begin{aligned} x_{it}^0 &= (\gamma_{it} + \eta_{it}\delta_{nt}) + (g_{it} + \eta_{it} e_{nt}) y_{t-1} \\ &\equiv \delta_{it} + e_{it} y_{t-1} , \qquad i = 1, \dots, n-1. \end{aligned} \tag{4.5}$$

Equations (4.4) and (4.5) can now be combined to give

$$x_t^0 = d_t + E_t y_{t-1} , \tag{4.6}$$

where

$$d_t = [\delta_{1t}, \dots, \delta_{nt}]'$$

and

$$E_t = [e'_{1t}, \dots, e'_{nt}]' ,$$

each δ_{it} being a scalar and e_{it} an m-dimensional row vector.

The formulas for how the coefficients of v_{it}, $i = 1, \dots, n$, are computed, given $v_{i,t+1}$ and (4.6) are the same as in Theorem 1.

In Figure 1 the points C and B are the solutions when player 1 and player 2, respectively, are dominant. The dominant player will choose a point such that the other player's reaction line is tangent to one of his own contour curves. In this particular example both players are better off in the dominant player solution than under noncooperative behavior. It is clear, then, that noncooperative solutions are not in general Pareto optimal.

We noted in Section 2 that if the two reaction lines are parallel, a noncooperative solution does not exist. It is clear, however, that if either of the players is dominant, one of his contours can still be tangent to the other's reaction line, and thus dominant player solutions may exist.

We saw in Section 2 that we could not dismiss the open loop solution completely as an equilibrium concept for the noncooperative model. In the dominant player model, however, the open loop solution has a very undesirable characteristic. Note first that the dominant player takes into account the reactions of the other players in all the T periods at once, and then announces his decisions for all periods. It might seem, then, that we have an equilibrium. However, this depends on the credibility of the dominant player. When the first period has elapsed, it pays for the dominant player not to stick to the original plan. If one resolves the problem for the remaining $T-1$ periods, the new first-period decisions will be different from the original plan for the second period.[13] If the players foresee this before the first period, they will choose the feedback solution instead. This solution has the desirable property that it is continually optimal through all the periods of the horizon. The dominant player then announces his first-period decision, taking into account the reaction functions of the other players, and with all the players correctly foreseeing the corresponding sequences of feedback equilibrium solutions in the future periods.[14]

The dominant player model can be generalized to more complicated hierarchical structures. We can think of all the players as divided into levels of dominance. The players on the highest level take into consideration the effects of their decisions on the decisions of the players on lower levels. The players on the lowest level take the decisions of the players on higher levels as given. The players on intermediate levels take the decisions of players on higher levels as given, while at the same time taking into consideration the effects of their decisions on the decisions of the players on lower levels. All the players on the same level are assumed to behave noncooperatively among themselves. The solution method would be a generalization of the method outlined in this section, where we have

[13] Continuing with the example of footnote 12, assume now that player 2 is dominant. The open loop decision rules are then

$$\begin{aligned} x_{11} &= -.6876y_{10} - .1127y_{20} + .2382 \\ x_{21} &= -.1127y_{10} - .6479y_{20} + .3239 \\ x_{12} &= -.1754y_{10} + .0141y_{20} + .0384 \\ x_{22} &= .0141y_{10} - .1690y_{20} + .0845 \end{aligned}$$

Assume again that $y_{10} = y_{20} = .1$. Then the solution is $x_{11} = .1581$, $x_{21} = .2479$, $x_{12} = .0223$, and $x_{22} = .0690$. However, if the problem is resolved after the first period, it turns out that the solution for the second period is $x_{12} = .0286$ and $x_{22} = .0500$, which for both players gives a higher value of the objective function for the last period than the original plan.

[14] In dominant player games we can also distinguish between closed loop and feedback solutions (see [25]). The closed loop solution is of feedback type but, the dominant player is assumed to announce his decisions for all periods at the start of the horizon. Like the open loop solution the closed loop solution is inconsistent under replanning.

only two levels of dominance, and one player on the highest level.

5. SOME REMARKS ON INFINITE HORIZON GAMES

The computational methods for finding the infinite period stationary feedback solutions by successive approximations would be similar to the methods outlined in Sections 2 and 4. It would take us too far to go into existence and convergence problems in this paper.[15] Suffice it here to say that for most problems convergence has turned out to be quite rapid. These stationary decision rules will be solutions to a set of interrelated functional equations, one for each player. Similarly, by letting T go to infinity in the open loop formulation, the first-period decision rule will converge to a stationary rule which in general will be different from the feedback stationary solution. The point we want to make is that for infinite horizon models we will discard the open loop solution as unsuitable for an equilibrium concept also for the noncooperative model. The reason is that if one player takes the stationary open loop rules of the other players as given and solves his infinite horizon one-player problem, his open loop decision rule is no longer the optimal stationary rule. However, if instead he takes the feedback decision rules of the other players as given and solves the one-player problem, his feedback decision rule will clearly be the optimal one.

Observe that when the optimal stationary decision rule, $x = d + Ey_{-1}$, is substituted into the moving equations, we get

$$y = (A + BE)y_{-1} + Bd + c$$

which leads to the following stationary solution

$$y^* = [I - (A + BE)]^{-1}(Bd + c)\,.$$

If we have a stochastic ε_t with covariance matrix Σ_ε independent of x, we find the mean $\bar{y}^*$ and the variance Σ_{y^*} of the stationary solutions as follows

$$\bar{y}^* = [I - (A + BE)]^{-1}(Bd + c)$$
$$\Sigma_{y^*} = (A + BE)\Sigma_{y^*}(A + BE)' + \Sigma_\varepsilon\,.$$

If in the last equations we define $A + BE = \|\alpha_{ij}\|$, $\Sigma_{y^*} = \|\sigma^y_{ij}\|$, and $\Sigma_\varepsilon = \|\sigma^\varepsilon_{ij}\|$, we can write

$$\sum_{l=1}^{m}\sum_{k=1}^{m}(\delta_{li}\delta_{kj} - \alpha_{il}\alpha_{jk})\sigma^y_{lk} = \sigma^\varepsilon_{ij}\,, \qquad \begin{aligned}\delta_{ij} &= 1 \text{ if } i = j,\\ \delta_{ij} &= 0 \text{ otherwise,}\\ i, j &= 1, \ldots, m.\end{aligned}$$

However, since Σ_{y^*} will be symmetric, we need to solve only $m(m-1)/2$ equations in the same number of unknowns to find Σ_{y^*}.

[15] Conditions for convergence that can be tested beforehand have not yet been established.

6. CONCLUDING COMMENTS

We have been concerned with finding suitable equilibrium solutions for non-cooperative and dominant player dynamic games. For that purpose we have compared open loop and feedback solutions, which in general are different, even in the absence of uncertainty. We argue that the feedback solution is generally more appropriate as an equilibrium concept. In the dominant player case the feedback solution is such that the plans of all the players are continually optimal throughout the horizon, while the open loop solution is inconsistent under replanning.

The potential applications of the results in this paper appear to be numerous. The introduction indicates that we originally had decentralized macroeconomic policymaking in mind. An application to a problem of that type was presented in Kydland **[14]**. In Kydland and Prescott **[15]** the problem of finding optimal stabilization policies for a competitive economy was formulated as a dominant player dynamic stochastic game. The policymaker is the dominant player, taking into account the reaction functions of economic agents. The results were found to have important implications for econometric policy evaluation.

A traditional application of game theory in economics has been oligopoly theory. In **[13]** we formulated a dynamic model for an oligopolistic industry in the framework of this paper. The initial results were sufficiently promising to lead one to believe that dynamic game-theoretic models may provide new insights into empirical phenomena in the area of industrial organization.

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