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HIERARCHICAL DECOMPOSITION IN LINEAR ECONOMIC MODELS*

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In decomposed linear programming models it is generally not possible to decentralize by prices alone. The Dantzig-Wolfe procedure, for instance, delegates weights on basic solutions in addition to the equilibrium prices. In this paper we present a decomposition procedure for linear models where we in addition to prices delegate a hierarchical ordering. In many problems this ordering makes the assignment of weights unnecessary, and gives the divisions more autonomy in their decision making. An operational condition is found for determining if, for any given problem, the new decomposition procedure will achieve coherent decentralization.

1. Introduction

The problem of finding equilibrium prices to delegate to the divisions of the economy insuring the overall optimum has been discussed by a long list of economists, including Koopmans ([10], [11]), Arrow-Hurwicz [1], Gale [8], Baumol-Fabian [3] and Charnes-Clower-Kortanek [5].[1] The difficulty is due to couplings or externalities among the divisions. They may use common resources so that the optimum of one division may preclude other divisions from reaching their optimum. The problem of reaching an equilibrium by prices alone has been particularly difficult in the case when the objective function and the constraints are all linear, and this is the case that will be treated in the present paper.[2]

The problems may arise at the microlevel in the organization of the firm, where the central unit, without knowing the feasible sets of the divisions, attempts to reach a global optimum only knowing the available overall resources. Decentralization also has important applications for national planning and resource allocation.[3]

It is generally possible to find prices such that the global optimum is also optimal for the divisions. These prices can for example be found using the Dantzig-Wolfe procedure [7].[4] With this method prices are given to the divisions by the central unit. The divisions then submit plans which may or may not satisfy the global constraints. A price-adjusting mechanism is then devised so that, after several iterations, the divisional plans bring the global plan as close to feasibility as possible, while producing a maximum increase in the value of the objective function.

An alternative method is due to Balas [2], where the purpose of the price-adjusting mechanism is to induce the divisions to submit such new plans that will bring the

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[1] Charnes-Clower-Kortanek refer to the problem as that of coherent decentralization.

[2] The problem of reaching equilibrium in linear systems was noted as far back as in 1949; see Samuelson [14].

[3] See articles by Kornai-Liptak [12] and Malinvaud [13].

[4] See also Dantzig [6, Chapter 23].

global constraints as close to feasibility as possible while preserving optimality in relation to the initial global objective function.

In both cases the solution with the final revised prices is usually not unique, and the divisions cannot find the correct one among the infinitely many alternative solutions without further information. The final step will generally involve delegating not only the final prices, but also weights for the divisions to use on solutions from previous iterations, which means that the central unit essentially tells the divisions what action to take, and the prices may have little importance at the end.

Charnes-Clower-Kortanek [5] proposed an alternative type of information, namely preemptive goals, which in addition to prices will lead to coherent decentralization. This essentially means delegating part of the overall resources to each of the divisions in the form of goals with penalties for deviations, such that if the goals are met, the requirement of global feasibility for the solutions will also be met. This procedure thus requires a lot of information to be transmitted from the central unit to the divisions, but it does give the divisions more autonomy in the decision process than the ones mentioned so far.

The purpose of this paper is to explore an alternative form of information that may be used in many cases, and which gives the divisions more autonomy than the previous procedures. This is based on the idea of a hierarchical ordering; that is, there may either be a natural order in which the decisions among the divisions will be carried out, or else an ordering may be imposed on them.[5]

In §2 we define a hierarchical ordering and hierarchical decomposition, and prove some results showing how coherent decentralization can be achieved by means of the hierarchical ordering. In §3 we indicate potential applications in central planning, and give illustrating numerical examples in §4. Finally, some concluding comments are offered in §5.

2. Hierarchical Decomposition

Consider the hierarchical master problem

$$\text{(M)}\qquad \begin{aligned} &\text{maximize} && \textstyle\sum_{j=1}^{p} c_j x_j \\ &\text{s. t.} && A_j x_j = d_j, && j = 1, \dots, p, \\ &&& \textstyle\sum_{k=1}^{j} C_{jk} x_k = d_{0j}, && j = 2, \dots, p, \\ &&& x_j \geqq 0, && j = 1, \dots, p, \end{aligned}$$

where A_j is $m_j \times n_j$, $j = 1, \dots, p$, and C_{jk} is $m_{0j} \times n_k$, $j = 2, \dots, p$.

Assume that none of the constraints of (M) are redundant. The global constraints are assumed to have an almost lower block-triangular structure. Note that, possibly after rearranging the order of the divisions, any decomposition problem will fit into the general form of (M). However, in practice we will often have $m_{0j} = 0$ for several j, perhaps even for all $j < p$.

The dual to (M) is

$$\text{(D)}\qquad \begin{aligned} &\text{minimize} && \textstyle\sum_{j=1}^{p} u_j d_j + \sum_{j=2}^{p} v_j d_{0j}, \\ &\text{s. t.} && u_j A_j + \textstyle\sum_{k=\max\{2,j\}}^{p} v_k C_{kj} \geqq c_j, && j = 1, \dots, p. \end{aligned}$$

Let the optimal solution to (D) be $(\bar{u}_1, \dots, \bar{u}_p, \bar{v}_2, \dots, \bar{v}_p)$.

[5] The issue of causal ordering has been thoroughly discussed by Simon [15].

Consider the following divisional problems:

Division 1 (to be solved first)

$$\text{(P}_1\text{)} \qquad \begin{aligned} &\text{maximize} \quad (c_1 - \textstyle\sum_{i=2}^{p} \bar{v}_i C_{i1}) x_1 , \\ &\text{s. t.} \qquad A_1 x_1 = d_1 , \qquad x_1 \geqq 0, \end{aligned}$$

with dual problem

$$\text{(D}_1\text{)} \qquad \text{minimize} \quad u_1 d_1 , \qquad \text{s. t.} \quad u_1 A_1 \geqq c_1 - \textstyle\sum_{i=2}^{p} \bar{v}_i C_{i1} .$$

Let the optimal solutions to the primal and dual problems be x_1^* and u_1^*, respectively.

Division k, $1 < k < p$ (to be solved after the solutions $x_1^*, \ldots, x_{k-1}^*$ to the previous divisions have been obtained)

$$\text{(P}_k\text{)} \qquad \begin{aligned} &\text{maximize} \quad (c_k - \textstyle\sum_{i=k+1}^{p} \bar{v}_i C_{ik}) x_k , \\ &\text{s. t.} \qquad A_k x_k = d_k , \\ &\qquad\qquad C_{kk} x_k = d_{0k} - \textstyle\sum_{j=1}^{k-1} C_{kj} x_j^*, \qquad x_k \geqq 0, \end{aligned}$$

with dual problem

$$\text{(D}_k\text{)} \qquad \begin{aligned} &\text{minimize} \quad u_k d_k + v_k (d_{0k} - \textstyle\sum_{j=1}^{k-1} C_{kj} x_j^*) \\ &\text{s. t.} \qquad u_k A_k + v_k C_{kk} \geqq c_k - \textstyle\sum_{i=k+1}^{p} \bar{v}_i C_{ik} . \end{aligned}$$

Let the optimal solutions to the above problems be x_k^* and (u_k^*, v_k^*), respectively.

Division p (to be solved after the solutions $x_1^*, \ldots, x_{p-1}^*$ have been obtained)

$$\text{(P}_p\text{)} \qquad \begin{aligned} &\text{maximize} \quad c_p x_p , \qquad \text{s. t.} \quad A_p x_p = d_p \\ &\qquad C_{pp} x_p = d_{0p} - \textstyle\sum_{j=1}^{p-1} C_{pj} x_j^*, \qquad x_p \geqq 0, \end{aligned}$$

with dual problem

$$\text{(D}_p\text{)} \qquad \begin{aligned} &\text{minimize} \quad u_p d_p + v_p (d_{0p} - \textstyle\sum_{j=1}^{p-1} C_{pj} x_j^*) \\ &\text{s. t.} \qquad u_p A_p + v_p C_{pp} \geqq c_p . \end{aligned}$$

Definition 1. By a *hierarchical ordering* we shall mean an ordering of the divisions such that the formulations (P_j), $j = 1, \ldots, p$, of the divisional problems are possible, with division j receiving information of how much is remaining of the overall resource vector d_{0j} after divisions $1, \ldots, j - 1$ have solved their problems.

Definition 2. *Hierarchical decomposition* means delegating to the divisions a hierarchical ordering along with the prices defined by the objective functions of (P_j), $j = 1, \ldots, p$.

We note that the revised prices defined here are slightly different from the ones used by Dantzig-Wolfe. While Dantzig-Wolfe correct for all of the global resources used, we do not charge anything for the amount that division j uses of the resources d_{0j} of which it is about to use up the remainder after divisions $1, \ldots, j - 1$ have determined how much they want.

First we shall prove the following theorem giving some insight into the problem.[6]

THEOREM 1. *Let the optimal solution to the master problem* (M) *be* $(\bar{x}_1, \ldots, \bar{x}_p)$. *Then, assuming that* $\bar{x}_i$ *is feasible for the revised divisional problem* (P_i) *for all* i, *it follows that* $\bar{x}_i$ *is optimal for the divisions for all* i.

[6] I am indebted to Sten Thore for the idea to this theorem. See also Thore-Kydland [16].

PROOF. $\bar{x}_i$ is feasible to (P_1) by standard decomposition theory. For $1 < k \leqq p$ we have by assumption that $\bar{x}_k$ is (P_k)-feasible.[7] This implies that

$$C_{kk}\bar{x}_k = d_{0k} - \sum_{j=1}^{k-1} C_{kj}x_j^*. \tag{1}$$

But from (M) we have

$$C_{kk}\bar{x}_k = d_{0k} - \sum_{j=1}^{k-1} C_{kj}\bar{x}_j\,. \tag{2}$$

Together (1) and (2) imply

$$\sum_{j=1}^{k-1} C_{kj}x_j^* = \sum_{j=1}^{k-1} C_{kj}\bar{x}_j\,. \tag{3}$$

Turning to the dual side, note that $\bar{u}_1$ is (D_1)—feasible, and $(\bar{u}_k, \bar{v}_k)$ is (D_k)—feasible, $k = 2, \ldots, p$. We therefore have

$$(c_1 - \sum_{i=2}^{p} \bar{v}_i C_{i1})\bar{x}_1 \leqq \bar{u}_1 d_1\,,$$

$$\begin{aligned}(c_k - \sum_{i=k+1}^{p} \bar{v}_i C_{ik})\bar{x}_k &\leqq \bar{u}_k d_k + \bar{v}_k(d_{0k} - \sum_{j=1}^{k-1} C_{kj}x_j^*) \\ &= \bar{u}_k d_k + \bar{v}_k(d_{0k} - \sum_{j=1}^{k-1} C_{kj}\bar{x}_j), \qquad k = 2, \ldots, p,\end{aligned} \tag{4}$$

where the last equality follows from (3). Summing these inequalities we get:

$$\begin{aligned}&\sum_{j=1}^{p} c_j\bar{x}_j - \sum_{k=1}^{p}\sum_{i=k+1}^{p} \bar{v}_i C_{ik}\bar{x}_k \\ &\qquad \leqq \sum_{k=1}^{p} \bar{u}_k d_k + \sum_{k=2}^{p} \bar{v}_k d_{0k} - \sum_{k=2}^{p}\sum_{j=1}^{k-1} \bar{v}_k C_{kj}\bar{x}_j \\ &\Leftrightarrow \sum_{k=1}^{p} c_k\bar{x}_k - \sum_{k=1}^{p-1}\sum_{i=k+1}^{p} \bar{v}_i C_{ik}\bar{x}_k \\ &\qquad \leqq \sum_{k=1}^{p} \bar{u}_k d_k + \sum_{k=2}^{p} \bar{v}_k d_{0k} - \sum_{j=1}^{p-1}\sum_{k=j+1}^{p} \bar{v}_k C_{kj}\bar{x}_j \\ &\Leftrightarrow \sum_{j=1}^{p} c_k\bar{x}_k \leqq \sum_{k=1}^{p} \bar{u}_k d_k + \sum_{k=2}^{p} \bar{v}_k d_{0k}\,.\end{aligned}$$

But since $(\bar{x}_1, \ldots, \bar{x}_p)$ and $(\bar{u}_1, \ldots, \bar{u}_p, \bar{v}_2, \ldots, \bar{v}_p)$ are optimal to (M) and (D), respectively, the above inequality must hold with equality. Therefore, the inequalities in (4) must hold with equalities. Q.E.D.

The main results of this paper are the following theorems.

THEOREM 2. *Let the optimal solution to* (M) *be* $(\bar{x}_1, \ldots, \bar{x}_p)$. *Then* $\bar{x}_1$ *is an optimal solution to* (P_1). *Taking each division* i, $i = 2, \ldots, p$, *in succession, we have that if, for all* i, $\bar{x}_{i-1}$ *is a unique optimal solution to* (P_{i-1}), *then* $\bar{x}_i$ *is optimal to* (P_i).

PROOF. The problem formulation (P_1) for division 1 is the same as in the Dantzig-Wolfe formulation. It is well known that a master optimum will always constitute optima for the divisions using the revised prices for common resources according to Dantzig-Wolfe. If this optimum for division 1 is unique, then $x_1^* = \bar{x}_1$, $u_1^* = \bar{u}_1$.

In general, let us look at division k, $1 < k \leqq p$, and assume that $x_1^*, \ldots, x_{k-1}^*$ have been successively obtained, and have all been unique for the respective divisions. Forming the divisional problem for division k in the standard manner (as opposed to the formulation (P_k)) we get maximize $(c_k - \sum_{i=k}^{p} \bar{v}_i C_{ik})x_k$ subject to $A_k x_k = d_k$, $x_k \geqq 0$, and its dual problem: minimize $u_k d_k$ subject to $u_k A_k \geqq c_k - \sum_{i=k}^{p} \bar{v}_i C_{ik}$. The master optimum $\bar{x}_k$ must constitute an optimum for this division, so that $(c_k - \sum_{i=k}^{p} \bar{v}_i C_{ik})\bar{x}_k = \bar{u}_k d_k$. Rearranging, we get

$$(c_k - \sum_{i=k+1}^{p} \bar{v}_i C_{ik})\bar{x}_k = \bar{u}_k d_k + \bar{v}_k C_{kk}\bar{x}_k\,. \tag{5}$$

However, from (M), and since $x_j^* = \bar{x}_j$, $j = 1, \ldots, k-1$, $C_{kk}x_k = d_{0k} -$

[7] The proof for $k = p$ is the same as for $1 < k < p$ if we define $\sum_{i=p+1}^{p} \equiv 0$. This is the case also in Theorem 2.

$\sum_{j=1}^{k-1} C_{kj}\bar{x}_j = d_{0k} - \sum_{j=1}^{k-1} C_{kj}x_j^*$ which, inserted into (5), gives

$$(c_k - \textstyle\sum_{i=k+1}^{p} \bar{v}_i C_{ik})\bar{x}_k = \bar{u}_k d_k + \bar{v}_k(d_{0k} - \textstyle\sum_{j=1}^{k-1} C_{kj}x_j^*) \tag{6}$$

implying that $\bar{x}_k$ is optimal for (P_k).

If this optimum is unique, then $x_k^* = \bar{x}_k$, $u_k^* = \bar{u}_k$, $v_k^* = \bar{v}_k$. Q.E.D.

In Theorem 2 we require the optimal solution for each division to be unique. It would be helpful to know when this is the case, and that is the subject of the following theorem.[8]

THEOREM 3. *Let q_j be the number of basic variables in $\bar{x}_j$ relative to the global problem, and let $r_j = q_j - m_j$, $j = 1, \ldots, p$. If $m_{0j} = r_j$ for all j when hierarchical decomposition is used, then, for the divisional optimal solutions to constitute the global solution, it is sufficient to delegate the hierarchical ordering in addition to a set of prices.*

PROOF. The number of basic variables in a linear programming problem is equal to the number of constraints (excluding the nonnegativity constraints). Therefore, if $m_{0j} = r_j = 0$, that is, if the number of basic variables in $\bar{x}_j$ is equal to the number of rows in A_j, then $\bar{x}_j$ is a basic feasible solution for the constraint set $\{A_j x_j = d_j, x_j \geqq 0\}$. In this case the Dantzig-Wolfe procedure provides a set of prices such that $\bar{x}_j$ is chosen as optimal by division j.

If $r_j > 0$, then the basic variables of $\bar{x}_j$ relative to the global problem are not a basic solution for the set of equations $A_j x_j = d_j$. However, if we add the m_{0j} equations $C_{jj}x_j = d_{0j} - \sum_{i=1}^{j-1} C_{ji}x_i^*$, and if $m_{0j} = r_j$, then $\bar{x}_j$ is a basic solution to this new system of $q_j = m_j + m_{0j}$ equations if the columns of the matrix $[A'_j : C'_{jj}]'$ relative to the basis elements of $\bar{x}_j$ form a basis of rank q_j for division j. Denote this potential basis by B_j, and the basis corresponding to $\bar{x}$ for the global problem by B.

To show that B_j is nonsingular, we expand $|B|$ and the resulting minors successively in such a way that we are left with a sum of products, each product containing $|B_j|$ as a factor. Assume that the column vectors of B are ordered by division in increasing order. Then the expansion of $|B|$ can be done in the following way. First we expand B by its first row (unless $j = 1$). This gives us a sum of products of the coefficients from the first row of B_1 with their cofactors in B. Now we expand the resulting minors by their first row, that is, by elements originally from the second row of B, and so on until the first row of each minor contains elements from the first row of B_j. Then we expand the remaining minors successively by all of the rows with coefficients from C_{ik}, $i = 1, \ldots, j - 1$, and all k. Because of the nearly lower block triangular structure of the global constraints, this last expansion will leave all of B_j in the resulting minors. Finally, we expand all of these minors by all the columns corresponding to divisions $j + 1, \ldots, p$. That leaves us with the desired expanded form of $|B|$ containing no other minors than $|B_j|$.

In order for $|B|$ to be nonzero we must have $|B_j| \neq 0$. B_j represents a basis for division j yielding $\bar{x}_j$ as a basic feasible solution. It is well known that every basic feasible solution corresponds to an extreme point of the set of solutions,[9] and that, for every extreme point, there is a price vector such that this point is the unique optimal solution.[10] Q.E.D.

[8] The dimensions m_j and m_{0j} were specified in problem formulation (M). We define m_{01} to be zero.

[9] See Gass [9, p. 52].

[10] If the price vector as defined by (P_j) leads to alternative optima, we know that there is an arbitrarily small perturbation of the price vector that makes the desired solution the only optimal one. See also [5, p. 314].

It is clear that we could not in general have $m_{0j} < r_j$, because then the rank of any basis for division j would be less than q_j. However, the reason for excluding the case $m_{0j} > r_j$ possibly needs a justification.

Let $m = \sum_{j=1}^{p} (m_j + m_{0j})$. We know that $\sum_{j=1}^{p} q_j = m$. We also have that $\sum_{j=1}^{p} r_j = \sum_{j=1}^{p} q_j - \sum_{j=1}^{p} m_j = \sum_{j=1}^{p} m_{0j}$. If $m_{0j} > r_j$ for some j, then there is a division k for which $m_{0k} < r_k$, which in general is not permissible.

In cases of degeneracy, where the number of nonzero variables is less than the number of basic variables, it may still be possible to find prices to insure coordinated action with $m_{0j} < r_j$ for some j as long as the number of nonzero variables is not larger than $m_j + m_{0j}$.

Whether the conditions for Theorem 3 are satisfied or not depends on the existence of a nearly lower block triangular structure in the global constraints. However, if the condition $m_{0j} = r_j$ is not satisfied for all the divisions, we can still use hierarchical decomposition supplemented by $r_j - m_{0j}$ preemptive goals for division j. The most extreme case of an absence of a lower block triangular structure would be the one where $m_{0j} = 0$ for all $j = 1, \ldots, p - 1$. In that case we would need to delegate r_j goals to all but the last division.

It is interesting at this point to compare our results with those of Charnes-Clower-Kortanek [5]. Their method of preemptive goals is equivalently stated in terms of goals via the objective function or via the constraints. The latter method is more easily comparable with ours. For simplicity we write the global constraints as $\sum_{k=1}^{p} C_k x_k = d_0$, that is, we do not partition the rows. Charnes-Clower-Kortanek delegate to each division j the quantities $\alpha_j = C_j \bar{x}_j$ of the overall resources along with prices according to Dantzig-Wolfe. This means delegating $\sum_{j=1}^{p} m_{0j}$ preemptive goals to each division. Our results show that no more than r_j goals are needed.

3. Application in Central Planning

It may be interesting to see how hierarchical decomposition can be used in central planning of an organization in a similar fashion as the Dantzig-Wolfe method. But now, instead of assigning weights at the end, we take advantage of a small piece of information which can be obtained from the plans that the divisions submit to the central unit. When the divisional problems are nondegenerate, the number of nonzero variables in their plans indicates how many constraints they are working with and is denoted by m_j here. When the central unit has determined the optimal values for the dual variables, but before the duals are used in computing the final revised prices, the central unit can, for each division j, subtract m_j from the number of basic variables in the part of the global solution relative to division j. The result is what we have denoted by r_j. For divisions with $r_j = 0$ there is no difficulty. For them prices are sufficient to induce them to choose their part of the optimal global plan.

For the divisions with $r_j > 0$ the central unit will try to find an ordering such that the remaining columns, if the coupling constraints are properly reordered, form a lower block-triangular matrix with $m_{0j} = r_j$; that is, the part of the lower block-triangular matrix with only zero-blocks for divisions $j + 1, \ldots, p$ has a row dimension of r_j. If this is possible, it is sufficient for division j to be given a set of prices and to know how much remains of the overall resource vector d_{0j} after the previous divisions in the ordering have solved their problems.

If it is impossible to find a complete hierarchical ordering that works, it may still be possible to find a partial one; that is, for some divisions to whom weights have to be assigned under the Dantzig-Wolfe procedure, we can instead use the hierarchical ordering procedure, while the remaining divisions may be given $r_j - m_{0j}$ preemptive goals. This means that at least for some divisions an increase in autonomy may take place.

A hierarchical ordering seems particularly natural in the case where most of the global constraints are of the pass-over type with coefficient $+1$ in some column for one of the divisions, -1 for some other division, and with right-hand side equal to zero. An example of this is when the product of one division is used as an input in the production of some other division. One of these divisions can then determine what quantity should be delivered, and the theory of hierarchical decomposition provides a way for the central unit to find out whether the delivering or receiving division should have this role.

4. Numerical Examples

1. The first example to be presented is described in detail in [6]. Here we will only formulate the problem and see how the theory of the previous sections can be applied. The problem is presented in Table 1.

The optimal global solution is $\bar{x}'_1 = (7.13,\ 64.13,\ 0,\ 10.87)$, $\bar{x}'_2 = (17.24,\ 8.12,\ 87.19,\ 7.13,\ 0,\ 0)$ and $\bar{x}'_3 = (1.06,\ 7.06,\ 8.12,\ 0,\ 92.94)$. The duals relative to the two coupling constraints are 2.94 and 2.83 respectively.

We see immediately that $r_1 = 0$, $r_2 = 1$, and $r_3 = 1$. This means that if we delegate the standard corrected prices $(6.95, 8, 0, 0)$ to division 1, $(3, 7.83, 8, -6.94, 0, 0)$ to division 2, and $(0, 9, -7.83, 0, 0)$ to division 3, then division 1 will find $\bar{x}_1$ as its optimal solution, while $\bar{x}_2$ and $\bar{x}_3$ are strict convex combinations of basic solutions for the respective divisions.

Fortunately, it turns out that in the problem above a complete hierarchical ordering can be assigned instead of weights. The ordering is the one given above, and we have $m_{02} = r_2 = 1$ and $m_{03} = r_3 = 1$, satisfying the condition in Theorem 3. Division 1 must solve its problem first and will get $x_1^* = \bar{x}_1$. Then division 2, knowing x_{11}^*, will solve the problem

$$\begin{aligned}
\max \quad & 3x_{21} + 7.83x_{22} + 8x_{23} - 4x_{24} \\
\text{s. t.} \quad & x_{21} + x_{22} + x_{23} - x_{24} = 105.42, \\
& -x_{21} + 0.2x_{24} + x_{25} = -15.81, \\
& -1.5x_{22} + x_{23} + x_{26} = 75, \\
& x_{24} = x_{11}^* = 7.13, \qquad x\text{'s} \geqq 0,
\end{aligned}$$

which has the unique optimal solution $x_2^* = \bar{x}_2$.

Finally, knowing x_{22}^*, division 3 will solve

$$\begin{aligned}
\max \quad & 9x_{32} - 5x_{33} \\
\text{s. t.} \quad & x_{31} + x_{32} - x_{33} = 0, \\
& -x_{31} + 0.15x_{32} + x_{34} = 0, \\
& x_{32} + x_{35} = 100, \\
& x_{33} = x_{22}^* = 8.12, \qquad x\text{'s} \geqq 0,
\end{aligned}$$

with the unique optimal solution $x_3^* = \bar{x}_3$.

TABLE 1

	Division 1				Division 2						Division 3					
Variable	x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}	x_{25}	x_{26}	x_{31}	x_{32}	x_{33}	x_{34}	x_{35}	
Profit	4	8	0	0	3	5	8	−4	0	0	0	9	−5	0	0	
Divisional constraints	1	1														= 71.26
	−1		1													= −7.126
		1		1												= 75
					1	1	1	−1								= 105.42
					−1			0.2	1							= −15.81
						−1.5	1			1						= 75
											1	1	−1			= 0
											−1	0.15		1		= 0
												1			1	= 100
Coupling constraints	−1							1								= 0
						−1							1			= 0

2. Our second example is the Birch Paper Company case [4], which was also used by Charnes-Clower-Kortanek [5]. The Birch Paper Company consists of three divisions N, S, and T, and the constraints are as follows[11]

TABLE 2

Variable	u_T	u_W	u_E	u_{NT}	u_{ET}	u_{TS}	u_{ES}	
Costs	0.48	0.43	0.432	−0.08	−0.005	−0.112	−0.036	
	1	1	1					$\geqq 1000$
				−1	−1			$\geqq -a_T$
						−1	−1	$\geqq -a_S$
	1			−1				= 0
	1					−1		= 0
			1		−1			= 0
			1				−1	= 0

We shall use the second of the examples in [5], namely where $a_T = 700$ and $a_S = 600$. The global optimal solution, including one slack variable as the last variable for each division, is $\bar{x}'_N = (600, 400, 0, 0)$, $\bar{x}'_T = (600, 0, 100)$ and $\bar{x}'_S = (600, 0, 0)$. The dual solution is $(0.43, 0, 0.142, 0.08, -0.03, 0.005, -0.003)$,[12] and the three divisional problems with revised prices according to Dantzig-Wolfe become:

$$(N) \qquad \begin{array}{ll} \text{minimize} & 0.43u_T + 0.43u_W + 0.43u_E \\ \text{s. t.} & u_T + u_W + u_E \geqq 1000, \qquad u\text{'s} \geqq 0, \end{array}$$

$$(T) \qquad \begin{array}{ll} \text{minimize} & 0u_{NT} + 0u_{ET}\,, \\ \text{s. t.} & -u_{NT} - u_{ET} \geqq -700, \qquad u\text{'s} \geqq 0, \end{array}$$

and

$$(S) \qquad \begin{array}{ll} \text{minimize} & -0.142u_{TS} - 0.039u_{ES} \\ \text{s. t.} & -u_{TS} - u_{ES} \geqq -600, \qquad u\text{'s} \geqq 0. \end{array}$$

We see that the global solution is optimal for all the divisions, but only S can find this solution without further information, such as weights on basic solutions.

Now let us try to find a hierarchical ordering which will lead the divisions to the global solution without delegating weights or preemptive goals. The optimal solution is degenerate, and therefore there is more than one basis that can correspond to this solution. One of these is such that $r_N = 2$, $r_T = 2$ and $r_S = 0$. This means that S must solve its problem first, giving the solution $x_S^* = \bar{x}_S$. We must then order N second and T third. The second and fourth coupling constraints give $m_{0N} = 2 = r_N$. N, knowing u_{TS}^* and u_{ES}^* from S, is given the prices $(0.40, 0.43, 0.427)$, and his optimal solution is now $x_N^* = \bar{x}_N$. Similarly, the first and third global constraints give $m_{0T} = 2 = r_T$. Division T, knowing u_T^* and u_E^* from N, is given the prices $(-0.08, -0.005)$ and obtains $x_T^* = \bar{x}_T$.

This problem illustrates the fact that for degenerate problems the condition of

[11] See [5] for a description of the problem.

[12] Note that the dual solution for this example, and therefore also the revised prices in the divisional problems, is wrong in [5].

Theorem 3 may be stronger than necessary. Both division T and N need to know the right-hand side of only one additional constraint instead of the two required by the theorem.

3. Finally we shall give a simple example where hierarchical decomposition does not work. Consider the problem

$$\begin{aligned} \max \quad & x_1 + x_2, \\ \text{s. t.} \quad & x_1 \leqq 1, \qquad x_2 \leqq 1, \\ & 2x_1 + x_2 \leqq \tfrac{5}{2}, \quad x_1 + 2x_2 \leqq \tfrac{5}{2}, \quad x_1, x_2 \geqq 0. \end{aligned}$$

Introducing slack variables and solving, we find that $\bar{x}_1 = \bar{x}_2 = \frac{5}{6}$. If the first two constraints are seen as divisional constraints and the last two as global constraints, it is clear that for both of the two possible orderings we have $r_1 = 1$, while m_{01} is defined as zero. However, if instead the objective function is $3x_1 + x_2$, then the global solution is $\bar{x}_1 = 1$ and $x_2 = \frac{1}{2}$. With division 1 solving first, we now have $r_1 = 0 = m_{01}$ and $r_2 = 2 = m_{02}$ and we can therefore use hierarchical decomposition.

5. Conclusion

In this paper we have formulated the decomposition problem for linear programming in a particular way that is convenient for the theory of hierarchical decomposition. This formulation leads to a specification of the divisional problems that is somewhat different from any one presented in the literature so far, both in terms of how the prices are revised, and also in terms of additional constraints that each division may take account of. We define a hierarchical ordering which, when delegated along with the revised prices, under conditions given in Theorem 3 insures coherent decentralization.

We have stressed the importance of leaving much autonomy with the divisions in making the final decision. The Dantzig-Wolfe procedure is deficient in this respect in that it implies that some divisions must be told by the central unit what action to take. The method of preemptive goals by Charnes-Clower-Kortanek was presented as a way of giving the divisions more autonomy in making the final decision, although there may not be much choice left when all the preemptive goals have been specified. With the hierarchical decomposition procedure each division is free to make the final decision and use whatever resources are optimal to him given what has been used by the previous divisions in the ordering. This leaves the divisions with at least as much autonomy as under the method of preemptive goals. For situations where the hierarchical ordering has to be supplemented by, for instance, preemptive goals, we provide the central unit with a rule for deciding how many goals are needed. As a corollary we found that the method of Charnes-Clower-Kortanek as it stands means delegating more information than is necessary, and we have sharpened their results in that regard. In the special case of pass-over constraints we pointed out in §3 that it is not even necessary to go via the central unit. Instead the information on quantities can be passed on directly from one division to another.

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