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Equilibrium Solutions in Dynamic Dominant-Player Models

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1. INTRODUCTION

A solution concept from game theory which has been used a lot in economic applications is the noncooperative solution or Nash equilibrium [10]. While much of the theory is concerned with static models, there has recently been an increasing interest in dynamic game-theoretic models. Noncooperative equilibrium solutions to nonzero-sum discrete-time dynamic games were discussed in detail in [5], and several references to literature on differential games can be found there. Two different noncooperative solutions were discussed. The open-loop solution is a sequence of decisions for each time period, and these decisions all depend on the *initial* state, and, in the presence of uncertainty, on observed disturbances. In a recent paper, Brock [1] has studied open-loop solutions for a wide class of models. In the feedback (or closed-loop) solution, on the other hand, decision rules are determined for each time period as functions of the most recent state variables at that time. In [5] these solutions were evaluated as possible candidates for an equilibrium concept in economic models.

One area in which game theory has been extensively applied is oligopoly theory. Here too the models have been mainly static, or sequences of static models, for a large part in the spirit of Cournot [3]. Exceptions are [2, 11] in which dynamic models with structural interconnections over time were analyzed. Besides the purely noncooperative solution, there is another solution which has a long tradition in oligopoly theory, namely the dominant-firm solution dating back as far as Stackelberg's [13] book in 1934. When making its decision, the dominant firm takes account of the reactions of its rivals. Computations of feedback solutions for a dominant-player model with quadratic objective functions and linear constraints were outlined briefly in [5]. Possible applications of dominant-player models, other than in industrial organization, are models of macroeconomic stabilization in which the government may act as a dominant player.

In this paper we discuss dominant-player models in considerable detail. We shall be concerned with equilibrium solutions in the sense that, given the

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other players' decisions, no player will change his decision rule. We mentioned above that in the dynamic noncooperative games in [5] the closed-loop and feedback solutions[1] were the same. In the dominant-player case they will in general be different. We therefore have three possible candidates for an equilibrium concept, viz., open-loop, closed-loop, and feedback solutions. Both closed-loop and feedback decision rules are functions of the *current* state variables for the respective time periods. However, the closed-loop decision rules for the dominant player are assumed to be determined (and committed to) at time zero for all time periods of the horizon. The feedback solutions, on the other hand, can be thought of as being announced for the dominant player for one period at the time, given the state variables at that time, and rationally expecting equilibrium decisions to be made in future periods on the basis of the information at that time.

In Section 2 the three types of solutions are compared in fairly abstract terms from an equilibrium point of view. The difficulty with the open-loop and closed-loop solutions is that it is in general not optimal to carry through with the original plans. The feedback solution has the desirable characteristic that the plans are intertemporally consistent. In Section 3 we give details of the computations of dominant-player feedback solutions for linear–quadratic stochastic games. Extensions to more complicated hierarchical structures are indicated. Such structures may be of particular interest in oligopoly models.

The fact that the dominant player announces his decisions first is likely to mean that he has to base his decisions on less accurate information than the rivals can. For example, the actual values for the current period of some of the disturbances in the structural equations may be available only to the nondominant players, or the dominant player may have to base his decisions on preliminary data for the state variables that are subject to measurement errors which have been corrected by the time the nondominant players act. In Section 4 we outline solutions for a case like that, and find that some decision rules then turn out to be stochastic in some sense. The paper ends with some concluding comments in Section 5.

2. Dynamic Equilibrium Solutions

For the purpose of the general discussion of possible equilibrium concepts we shall use a somewhat more general and compact notation than what is used in later sections, although it is understood that the objective functions will be quadratic and the structural equations linear. For simplicity and

[1] A distinction between these two solution concepts is seldom made. See, however, Simaan and Cruz [12] and Tse [14].

without much loss of generality we assume that each player has control over one instrument. Then we can write the structural equations as follows:

$$y_t = f(y_{t-1}, x_t, \epsilon_t), \tag{1}$$

where y_t is an $m \times 1$ vector of state variables, x_t is an $n \times 1$ vector of decision variables or instruments, with $m > n$, and ϵ_t is a vector of disturbances which are independently distributed over time with mean zero and finite variances. Each player i wishes to minimize a preference (or loss) function

$$W_i = E\left[\sum_{t=1}^{T} w_{it}(y_t)\right], \qquad i = 1,..., n,$$

where E denotes an expectations operator. We assume that dependence of the objective functions on decision variables or changes thereof as well as the inclusion of lags in the structural equations have been taken care of by expanding the state vector as explained in [5].

Player i has control over x_i only, and he has to take into consideration what the other players do. The decision variables of the other players at time t will be denoted by $x_t^{(i)} = (x_{1t},..., x_{i-1,t}, x_{i+1,t},..., x_{nt})$. Player n will be the dominant player.

By an equilibrium solution we intuitively mean a solution such that, given what the other players do, no player will change his decision rule. We shall first look for solutions in policy space, with decision rules being sequences of the form $\{X_{it}(y_{t-1}, x_{nt})\}_{t=1}^T$, $i = 1,..., n - 1$, and $\{X_{nt}(y_{t-1})\}_{t=1}^T$.

DEFINITION. An equilibrium for each time period t, $t = 1,..., T$, is a set of decision rules $x_{it} = \bar{X}_{it}(y_{t-1}, x_{nt})$, $i = 1,..., n - 1$, and $x_{nt} = \bar{X}_{nt}(y_{t-1})$, such that

$$\min_{x_{it}} E[w_{it}(y_t) + v_{i,t+1}(y_t) | \bar{X}_t^{(i)}] = E[w_{it}(y_t) + v_{i,t+1}(y_t) | \bar{X}_t], \qquad i = 1,..., n,$$

where

$$v_{i,t+1}(y_t) = E\left[\sum_{s=t+1}^{T} w_{is}(y_s) | \, x_{js} = \bar{X}_{js}(y_{s-1}, x_{ns}), j = 1,..., n - 1, \right.$$
$$\left. x_{ns} = \bar{X}_{ns}(y_{s-1}), \, s = t + 1,..., T\right].$$

We note that in the definition above $v_{i,t+1}(y_t)$ is the total value for player i of the sequences of dominant-player solutions from time period $t + 1$ until the end of the horizon. The definition thus says that each player chooses the best decision rule for period t, given the last observed values of the state variables y_{t-1}, the decision rules of the other players, and that decisions will be similarly selected in periods $t + 1,..., T$.

We shall now outline how the solution for period t can be computed. As a first step, we find for $i = 1,\ldots, n-1$

$$V_{it}(y_{t-1}, x_t^{(i)}) = \min_{x_{it}} E[w_{it}(y_t) + v_{i,t+1}(y_t)] \tag{2}$$

subject to (1), and given y_{t-1} and $x_t^{(i)}$. The solution for each of the nondominant players (under conditions to be given in Section 3) is a function of y_{t-1} and $x_t^{(i)}$ derived from the first-order condition for a minimum. If these $n-1$ players behave noncooperatively among themselves, we can (again under conditions to be specified later) solve for

$$x_{it} = \bar{X}_{it}(y_{t-1}, x_{nt}), \qquad i = 1,\ldots, n-1. \tag{3}$$

For the dominant player the problem is to solve

$$v_{nt}(y_{t-1}) = \min_{x_{nt}} E[w_{nt}(y_t) + v_{n,t+1}(y_t)]$$

subject to (1) and (3), and given y_{t-1}. The optimal decision rule will be of the form

$$x_{nt} = \bar{X}_{nt}(y_{t-1}). \tag{4}$$

The decision rules given by (3) and (4) will clearly satisfy the definition of equilibrium. Finally, using (3) and (4) we can write V_{it} in (2) as

$$v_{it}(y_{t-1}) = V_{it}(y_{t-1}, \bar{X}_t^{(i)}), \qquad i = 1,\ldots, n-1.$$

The solution outlined above is the feedback solution. The decision rules are obtained by working backward recursively from period T until the initial period. An alternative would be for the dominant player to look at the horizon as a whole and determine the set of decision rules $\{X_{nt}^c(y_{t-1})\}_{t=1}^T$ that gives the lowest value of W_n while taking account of the rivals' reactions. This would be the closed-loop solution. However, even for the linear–quadratic case in which X_{nt}^c are linear, the computation of the closed-loop solution for a finite horizon is quite complicated and leads to a highly nonlinear problem. If the horizon is infinite, with $w_{it}(y_t) = \beta_i^{t-1} w_i(y_t)$, where β_i is a discount factor such that $0 < \beta_i < 1$, the decision rules will be the same for every period, and a search over the coefficients of the stationary $X_n{}^c(y_{t-1})$ can be carried out.

A third solution alternative is to look for the sequence $\{x_{nt}\}_{t=1}^T$ of values of the decision variables that minimizes W_n given the rivals' reactions. This sequence will depend on y_0 and, in the presence of uncertainty, on observed disturbances. We now outline briefly how this open-loop solution can be obtained.

To determine the reactions of the nondominant players we consider the problem for player i, $i = 1,..., n - 1$, which is

$$\underset{x_{i1},\dots,x_{iT}}{\text{minimize}}\; E\left[\sum_{t=1}^{T} w_{it}(y_t)\right]$$

subject to

$$y_t = f_t(y_0, x_1, \dots, x_t, \epsilon_1, \dots, \epsilon_t), \qquad t = 1, \dots, T, \tag{5}$$
$$y_0,\, x_1^{(i)}, \dots, x_T^{(i)} \text{ given.}$$

The functions f_t, $t = 1,..., T$, are obtained by successive iterations of the original function f.

From the first-order conditions, and assuming that the nondominant players behave noncooperatively among themselves, we get under appropriate conditions decision rules of the form

$$x_t^{(n)} = g_t^{(n)}(y_0, x_{n1}, \dots, x_{nT}, \epsilon_1, \dots, \epsilon_{t-1}), \qquad t = 1, \dots, T, \tag{6}$$

where $x_t^{(n)} = (x_{1t}, ..., x_{n-1,t})$, and correspondingly for $g_t^{(n)}$.

The problem for the dominant player is to

$$\underset{x_{n1},\dots,x_{nT}}{\text{minimize}}\; E\left[\sum_{t=1}^{T} w_{nt}(y_t)\right]$$

subject to (5) and (6), and given y_0. The solution to this minimization problem will be of the form

$$x_{nt} = g_{nt}(y_0, \epsilon_1, \dots, \epsilon_{t-1}), \qquad t = 1, \dots, T. \tag{7}$$

Thus, the open-loop decisions are given by (6) and (7).

Just as the feedback solution obviously satisfies our definition of equilibrium, it is equally apparent that the open-loop and the closed-loop solutions in general do not. One might ask at this point whether it is possible to think of definitions of equilibrium that either of these solutions would fit. This is clearly possible, and we therefore have to make an argument as to why this definition would be less likely as a good description of how an economic system would operate in practice.

Looking at the horizon as a whole, it is clear that both the open-loop and closed-loop solutions give lower value of the dominant-player loss function than the feedback solution does. Note that in all three solution concepts the decisions of the nondominant players at time t depend on what the dominant player is expected to do not only in period t, but also in periods $t + 1,..., T$. In the open-loop case this is clearly seen in Eq. (6). In the closed-loop and feedback solutions the coefficients of the decision rules $X_{it}(y_{t-1}, x_{nt})$ will depend on the expected decisions of the dominant player in periods $t+1,\ldots,T$. To make this clear, we could write $\bar{X}_{it}(y_{t-1}, x_{nt}; \bar{X}_{n,t+1},\ldots,\bar{X}_{nT})$ for the feedback case, and $X_{it}^c(y_{t-1}; X_{n1}^c,\ldots,X_{nT}^c)$ in the closed-loop case.

What makes the value of the solution inferior in the feedback case is the fact that the dominant player does not take into account at time t the effect his decision for that period t has on his rivals' decisions in periods $s<t$. To see this more clearly, assume that the problems can be written in terms of the decision variables only, and that there is no uncertainty. The dominant-player problem would be to

$$\underset{x_{n1},\ldots,x_{nT}}{\text{minimize}} \sum_{t=1}^{T} w_{nt}(x_t)$$

subject to

$$x_{it} = X_{it}(x_{n1},\ldots,x_{nT}),\ i=1,\ldots,n-1;\ t=1,\ldots,T.$$

The first-order conditions for a minimum are

$$\frac{\partial w_{nt}}{\partial x_{nt}} + \sum_{s=1}^{T}\sum_{i=1}^{n-1} \frac{\partial w_{ns}}{\partial x_{is}} \frac{\partial X_{is}}{\partial x_{nt}} = 0, \qquad t=1,\ldots,T. \tag{8}$$

However, if $x_1,\ldots,x_{t-1}$ are taken as given at time t, then the first-order conditions will be

$$\frac{\partial w_{nt}}{\partial x_{nt}} + \sum_{s=t}^{T}\sum_{i=1}^{n-1} \frac{\partial w_{ns}}{\partial x_{is}} \frac{\partial X_{is}}{\partial x_{nt}} = 0, \qquad t=1,\ldots,T.$$

The difference from (8) is

$$\sum_{s=1}^{t-1}\sum_{i=1}^{n-1} (\partial w_{ns}/\partial x_{is})(\partial X_{is}/\partial x_{nt}),$$

which in general will be different from zero. Only if $\partial X_{is}/\partial x_{nt} = 0$ for all $s<t$ could we be assured of this term being equal to zero.

We also realize something else from the demonstration above. If the problem is reevaluated for the remaining $T-k$ periods of the horizon after the first k periods have elapsed, we see that the conditions for an optimum from then on are

$$\frac{\partial w_{nt}}{\partial x_{nt}} + \sum_{s=k+1}^{T}\sum_{i=1}^{n-1} \frac{\partial w_{ns}}{\partial x_{is}} \frac{\partial X_{is}}{\partial x_{nt}} = 0, \qquad t=k+1,\ldots,T.$$

Again there is a difference from (8),

$$\sum_{s=1}^{k}\sum_{i=1}^{n-1} (\partial w_{ns}/\partial x_{is})(\partial X_{is}/\partial x_{nt}),$$

which in general will be different from zero. This means that the original plan is no longer optimal from that time on. Faced with this fact, one would expect a great temptation on the part of the dominant player to change his original plan. Only the feedback solution has the characteristic that the original plan will not be changed under replanning. Note that this argument does not depend on the presence of uncertainty in the model.

If the dominant player tries to carry out a closed-loop policy, say, but continually gets tempted to change his policy when the state changes, then there is no reason to get the expected outcome even in the first period. If the rivals come to expect policy changes, presumably this will affect their behavior from the start.

In practice economic agents are not likely to consciously carry out all the extremely complicated calculations required to find the solutions we have presented. Rather, the equilibrium solution is meant to indicate the path that an economic system would follow on the average, or converge toward as agents are learning more about how other agents behave, and correcting whatever errors they have made in the past. Thus a definition of equilibrium would not be of much use unless it implies a solution which is stable in the above sense. We have explained why the feedback solution is likely to be stable, while the open-loop and closed-loop solutions are not.

To sum up, there appear to be strong reasons to believe that the feedback solution is the one likely to give a good description of the movement of an economy with a dominant player. One might wonder if the open-loop or closed-loop solutions would provide a better description if the dominant player were the government. This does not seem likely, however, in particular not when an election draws near, or a new administration takes over. This comment suggests that a combination of the feedback solution with either open-loop or closed-loop solution for 3 or 4 years at the time might provide a reasonable description of how a government would operate.

Before giving a numerical example, we shall comment briefly on the infinite horizon problem. For this case we would presumably have $w_{it}(y_t) = \beta_i^{t-1} w_i(y_t)$, $i = 1,..., n$. The equilibrium feedback solutions would be stationary decision rules of the form $x_i = \bar{X}_i(y_{-1}, x_n)$ and $x_n = \bar{X}_n(y_{-1})$ satisfying the functional equations

$$V_i(y_{-1}, x_n) = \min_{x_i} E[w_i(y) + \beta_i v_i(y)], \qquad i = 1,..., n-1,$$

and

$$v_n(y_{-1}) = \min_{x_n} E[w_n(y) + \beta_n v_n(y) \mid x_i = \bar{X}_i(y_{-1}, x_n),\ i = 1,..., n-1],$$

with

$$v_i(y_{-1}) = V_i[y_{-1}, \bar{X}_n(y_{-1})], \qquad i = 1,..., n-1,$$

and all subject to $y = f(y_{-1}, x, \epsilon)$. For the linear–quadratic case, successive approximations, which are easily computable, have usually turned out to converge rather quickly.

For this case with stationary decision rules, assume that the players initially do not know for sure the decision rules of the other players. Using his best estimate each player computes the best decision rule, given what he thinks the others will do. When the actual decision rules turn out different than expected, the expectations are revised. There are as usual several possibilities, the most naive of which is static expectations, which in this context means that the players expect the other players to behave in the future according to the last decision rule. Another possibility is adaptive expectations formation. If the system is stable, such a process will converge to equilibrium decision rules such that no player has any incentive to change his decision rule. From what was said above, it is clear that only the feedback solution could be stable in this sense. Such processes were discussed for the purely noncooperative case in [6], and in the context of a dominant-firm model in [7].

To illustrate some of the points of this section, a numerical example will be presented. Assume that the functions to be maximized are

$$W_i = \sum_{t=0}^{T} \left(\frac{1}{1+r}\right)^t [(1 - y_{1t} - y_{2t})\, y_{it} - q x_{it} - c(x_{it} - \delta y_{it})^2],$$

subject to $y_{i,t+1} = (1 - \delta)\, y_{it} + x_{it}$, $i = 1, 2$. This is a simplified version of an oligopoly model used in [7]. The variables x_i and y_i can be interpreted as investment and capital stock, respectively, for firm i. Output is constrained by capital stock, and units are chosen so that one unit of output requires one unit of capital. The industry faces a linear demand curve (which can be thought of as being adjusted for any constant unit production costs). There are constant returns to scale in the long run, but changes in capacity are subject to increasing cost of adjustment. Future profits are discounted using the interest rate r. The rate of depreciation is denoted by δ.

In the present example we shall use the values $r = 0.1$, $\delta = 0.1$, $q = 1$, and $c = 2$. The horizon T is chosen long enough for the solutions to approximate closely those of an infinite horizon. In comparing the three types of solutions we shall concentrate on the decision rules and profits of the dominant firm (firm 2), the stationary capital stocks, and whether or not these stocks are stable under replanning. In order to make these comparisons, we also need initial values of the capital stocks. We shall use $y_0{}^A = (0, 0.4)$, meaning that firm 1 is just entering an industry in which firm 2 had a monopoly, or, alternatively, $y_0{}^B = (0.2, 0.4)$, which is the steady state for the open-loop solution, and also the solution of the static problem without cost of adjustment.

In Table I the total profits for the dominant firm are listed for the three solutions.

TABLE I

Initial state		Dominant-firm profits		
y_{10}	y_{20}	Feedback	Open loop	Closed loop
0	0.4	1.379	1.475	2.152
0.2	0.4	1.225	1.322	1.840

The feedback decision rule for the dominant firm is

$$x_{2t} = -0.1030y_{1t} - 0.3188y_{2t} + 0.1635$$

for all t, with the resulting steady state $y_*{}^f = (0.2633, 0.3257)$.

In the open-loop case the first-period decision as a function of y_0 (the decisions for all periods are functions of y_0) is

$$x_{20} = -0.1004y_{10} - 0.3208y_{20} + 0.1683,$$

and the steady state is $y_*^0 = (0.2, 0.4)$. Note, however, that if we start out with $y_0 = y_*^0$, then y_{2t} will drop to 0.3799 in the first period and 0.3767 in the second period, with y_{1t} increasing to 0.2047 and 0.2073, for then slowly to move back toward y_*^0 again. Thus, the original plan is not optimal under replanning.

The dominant-firm closed-loop decision rule when the initial state is $y_0{}^A$ is

$$x_{2t} = 3.1y_{1t} + 0.13y_{2t} - 0.0385,$$

with steady state (0.0085, 0.4079). If the initial state is $y_0{}^B$, the closed-loop decision rule is

$$x_{2t} = 0.5y_{1t} + 0.13y_{2t} - 0.038,$$

with resulting steady state (0.0620, 0.2329). This illustrates the fact that the coefficients of the closed-loop decision rule, and also the resulting stationary solutions, depend on the initial state.

We note that the nature of the closed-loop decision rules is quite different from the other two solutions. Effectively, if the dominant firm threatens to meet any increases in the rival's capacity by substantial increases in his own capacity and the rival accepts this as given, then the dominant firm can force the rival to a very low market share. However, in practice it is unlikely that this would be the end of the story. For a more extensive analysis of this particular problem, see [7].

3. The Linear–Quadratic Feedback Solution

We now turn to the linear–quadratic case and shall assume that the loss functions can be written as[2]

$$w_i(y_t) = p_i'y_t + \tfrac{1}{2}y_t'Q_iy_t, \qquad i = 1,\dots, n,$$

and the structural equations as

$$y_t = Ay_{t-1} + Bx_t + c + \epsilon_t. \tag{9}$$

We assume that ϵ_t, $t = 1,\dots, T$, are identically and independently distributed over time with mean zero and finite covariance matrix Ω.

The value functions, which will turn out to be quadratic, will be written as

$$v_{it}(y_{t-1}) = v_{it} + r_{it}'y_{t-1} + \tfrac{1}{2}y_{t-1}'S_{it}y_{t-1}.$$

Define the notation:

$$H_t = \begin{bmatrix} b_1'\Sigma_{1t} \\ \vdots \\ b_{n-1}'\Sigma_{n-1,t} \end{bmatrix} \qquad \text{and} \qquad k_t = \begin{bmatrix} b_1'\rho_{1t} \\ \vdots \\ b_{n-1}'\rho_{n-1,t} \end{bmatrix},$$

where $\Sigma_{it} = Q_i + \beta_iS_{i,t+1}$, $\rho_{it} = p_i + \beta_ir_{i,t+1}$, and b_i is column i of the matrix B. Also, let $B^{(n)} = [b_1,\dots, b_{n-1}]$ and $x_t^{(n)} = (x_{1t},\dots, x_{n-1,t})'$. Thus, superscript (n) on a matrix or a vector means that the last column or the last element, respectively, have been deleted. Then we can prove the theorem:

Theorem 1. *Assume that*

(i) $b_i'\Sigma_{it}b_i > 0$, $i = 1,\dots, n-1$; $t = 1,\dots, T$,

(ii) $|H_tB^{(n)}| \neq 0$, $t = 1,\dots, T$,

(iii) $b_n'[I - B^{(n)}(H_tB^{(n)})^{-1}H_t]'\,\Sigma_{nt}[I - B^{(n)}(H_tB^{(n)})^{-1}H_t]\,b_n > 0,$
$t = 1,\dots, T.$

The unique equilibrium solutions for each period t, $t = 1,\dots, T$, can then be computed recursively:

$$x_t^{(n)} = G_ty_{t-1} + \eta_tx_{nt} + \gamma_t,$$

$$x_{nt} = d_{nt}y_{t-1} + \delta_{nt},$$

[2] A discussion of the generality of this formulation can be found in [5, p. 324].

where

$$G_t = -(H_tB^{(n)})^{-1} H_tA,$$

$$\eta_t = -(H_tB^{(n)})^{-1} H_tb_n ,$$

$$\gamma_t = -(H_tB^{(n)})^{-1} (H_tc + k_t),$$

$$d_{nt} = -\frac{(b_n + B^{(n)}\eta_t)' \Sigma_{nt}(A + B^{(n)}G_t)}{(b_n + B^{(n)}\eta_t)' \Sigma_{nt}(b_n + B^{(n)}\eta_t)},$$

$$\delta_{nt} = -\frac{(b_n + B^{(n)}\eta_t)' [\rho_{nt} + \Sigma_{nt}(c + B^{(n)}\gamma_t)]}{(b_n + B^{(n)}\eta_t)' \Sigma_{nt}(b_n + B^{(n)}\eta_t)} .$$

Writing the decision rules on the form

$$x_t = D_t y_{t-1} + \delta_t ,$$

where

$$D_t = \begin{bmatrix} G_t + \eta_t d_{nt} \\ d_{nt} \end{bmatrix} \quad \text{and} \quad \delta_t = \begin{bmatrix} \gamma_t + \eta_t \delta_{nt} \\ \delta_{nt} \end{bmatrix},$$

the coefficients of the value functions v_{it} for the n players are determined by the recursive relations

$$S_{it} = (A + BD_t)' \Sigma_{it}(A + BD_t),$$

$$r_{it} = (A + BD_t)' [\rho_{it} + \Sigma_{it}(B\delta_t + c)],$$

$$v_{it} = [\rho_{it} + \tfrac{1}{2}\Sigma_{it}(B\delta_t + c)]' (B\delta_t + c) + \tfrac{1}{2} \operatorname{trace} (\Sigma_{it}\Omega) + \beta_i v_{i,t+1} ,$$

with $S_{i,T+1}$, $r_{i,T+1}$, and $v_{i,T+1}$ all being zero.

Proof. It is easy to show that each v_{it} is quadratic if all $v_{i,t+1}$ are quadratic. The functions $v_{i,T+1}$ are the null function and therefore trivially quadratic, so all v_{it} are quadratic by induction.

Assume that $v_{i,t+1}$ has been found by backward induction for all i, $i = 1,..., n$. Then we can write

$$\begin{aligned} V_{it}(y_{t-1}, x_{nt}) &= \min_{x_{it}} E[w_i(y_t) + \beta_i v_{i,t+1}(y_t) | x_{nt}] \\ &= \min_{x_{it}} [\beta_i v_{i,t+1} + \rho_{it}'(Ay_{t-1} + Bx_t + c) \\ &\quad + \tfrac{1}{2}(Ay_{t-1} + Bx_t + c)' \Sigma_{it}(Ay_{t-1} + Bx_t + c) \\ &\quad + \tfrac{1}{2} \operatorname{trace} (\Sigma_{it}\Omega) | x_{nt}], \qquad i = 1,..., n-1. \end{aligned}$$

Differentiating the right-hand side, we get the first-order conditions:

$$b_i'\rho_{it} + b_i'\Sigma_{it}(Ay_{t-1} + Bx_t + c) = 0, \qquad i = 1,..., n-1.$$

By assumption (i) these conditions are also sufficient for a minimum for each of the players. By assumption (ii) we can solve this system of $n - 1$ equations for $x_t^{(n)}$ to get:

$$x_t^{(n)} = G_t y_{t-1} + \eta_t x_{nt} + \gamma_t , \tag{10}$$

where G_t, η_t, and γ_t are defined as in the theorem.

For the dominant player we write

$$\begin{aligned} v_{nt}(y_{t-1}) &= \min_{x_{nt}} E[w_n(y_t) + \beta_n v_{n,t+1}(y_t)] \\ &= \min_{x_{nt}} E[\beta_n v_{n,t+1} + \rho_{nt}' y_t + \tfrac{1}{2} y_t' \Sigma_{nt} y_t], \end{aligned}$$

subject to (10). A necessary condition for a minimum with respect to x_{nt} is:

$$\begin{aligned} &(b_n + B^{(n)}\eta_t)' \Sigma_{nt}[(A + B^{(n)} G_t) y_{t-1} + (b_n + B^{(n)}\eta_t) x_{nt} + c + B^{(n)}\gamma_t] \\ &\quad + (b_n + B^{(n)}\eta_t)' \rho_{nt} = 0. \end{aligned}$$

By assumption (iii) this condition is also sufficient for a minimum. Solving for x_{nt} we get:

$$x_{nt} = d_{nt} y_{t-1} + \delta_{nt} , \tag{11}$$

with d_{nt} and δ_{nt} defined as in the theorem. Substituting (11) into (10) we can write

$$x_t = D_t y_{t-1} + \delta_t , \tag{12}$$

with D_t and δ_t defined in the theorem. We can also write

$$v_{it}(y_{t-1}) = V_{it}(y_{t-1} , x_{nt} \mid x_{nt} = d_{nt} y_{t-1} + \delta_{nt}), \qquad i = 1,..., n - 1.$$

Substituting (12) into the value functions, we get both the left-hand sides and the right-hand sides in terms of y_{t-1}. Comparing the coefficients for the second-degree term, the first-degree term, and the constant, respectively, we get the recursive relations for S_{it}, r_{it}, and v_{it}, $i = 1,..., n$. This completes the proof.

Assumption (i) is rather weak because it can be satisfied for player i even if some diagonal elements of Q_i are negative, that is, Q_i need not even be positive semidefinite as is usually required in the standard control problem for one decision maker.

The implications of an assumption like (ii) have been thoroughly discussed in [5]. Assumption (iii), on the other hand, is special for the dominant-player problem. It can be shown that there are games, the payoff functions of which are quadratic with respect to the strategies, for which there is no equilibrium in the purely noncooperative case, but there may still be a dominant-player equilibrium. Moreover, the dominant-player solution always exists if only the payoff functions are negative definite with respect to the strategies.

For the infinite-horizon case the computations of successive approximations using value iterations would be similar to the computations given by the theorem, that is, going from one iteration to the next would be the same as going from period $t + 1$ to t in the theorem. Numerical examples have shown the successive approximations to converge rather quickly, in particular for the coefficients S_i of the quadratic parts, which are sufficient to determine D (or G, η, and d_n) regardless of r_i and ν_i, $i = 1,..., n$, and δ (or γ and δ_n). Note that each r_i is a linear function of δ, while δ is a linear function of r_i, $i = 1,..., n$. Computationally, this means that one can carry out the approximation only for S_i, $i = 1,..., n$, and D. Then the result can be used to determine r_i, $i = 1,..., n$, as linear functions of δ, which are then substituted into the expression for δ, thus determining δ, which again determines r_i, $i = 1,..., n$. Finally, simple calculations will give us ν_i, $i = 1,..., n$.

When the equilibrium stationary decision rule, $x = Dy_{-1} + \delta$, is substituted into the structural equations, we get

$$y = (A + BD)\, y_{-1} + B\delta + c + \epsilon,$$

which leads to the mean for the stationary solution:

$$\bar{y}^* = [I - (A + BD)]^{-1} (B\delta + c).$$

Given the assumption on ϵ, we can compute the covariance matrix of y^*, say Λ, by solving

$$\Lambda = (A + BD)\, \Lambda (A + BD)' + \Omega.$$

This equation can be solved for the symmetric matrix Λ as described in [5, p. 333].

In this paper we are restricting ourselves to the case of one dominant player and $n - 1$ noncooperative rivals. This particular assumption on the hierarchical structure could easily be relaxed. In general, we could have several hierarchical levels, each with a number of noncooperative players who take into account the reaction functions of the lower-level players while taking as given the decisions of the higher-level players. An interesting special case would be the one in which the n players all are on different levels. In an oligopoly, for instance, such a hierarchical structure would lead to an equilibrium in which the firms would all have different market shares in spite of having the same cost structure.

4. A Case of Partially Delayed Information

In some economic situations for which the dominant-player model may be appropriate, there may be a certain disadvantage to being dominant that is not taken account of in the solutions described in Sections 2 and 3. This disadvantage may be due to the fact that, for example, for institutional reasons, the dominant player has to make and announce his decision before certain data are known, while the same data may be known by the time the rivals make their decisions. For instance, in a model similar to the one described in [8] in which the government is dominant in determining the investment tax credit, while the private sector decides on how much to invest, the demand conditions for the period at hand may have become known by the time private firms make their decisions. Examples of economic models in which information lags are crucial, can be found in Cyert and DeGroot [4] and Lucas [9].

In this section, then, we shall outline the solution to a two-player linear-quadratic model in which the nondominant player knows the outcome of the disturbance ϵ_t when making the decision for period t, while the latest ϵ known to the dominant player is ϵ_{t-1}. The error term could, for instance, appear in an autoregressive equation determining a parameter of a demand curve.

Using for a moment the notation of Section 2, we can first explain in fairly general terms what will happen. The equilibrium (feedback) solutions for the two players will be two sequences of decision rules of the form $\{X_{1t}(y_{t-1}, x_{2t}, \epsilon_t)\}_{t=1}^{T}$ and $\{X_{2t}(y_{t-1})\}_{t=1}^{T}$, where player 2 is dominant. Corresponding to these decision rules we can define value functions v_{it}, $i = 1, 2$, that give the value for each player of following the decision rules from period t until the end of the horizon. These value functions will satisfy the functional equations:

$$v_{1t}(y_{t-1}, \epsilon_t) = \min_{x_{1t}} [w_1(y_t) + \beta_1 E_{t+1} v_{1,t+1}(y_t, \epsilon_{t+1}) \mid x_{2t} = X_{2t}(y_{t-1})],$$

where E_{t+1} denotes an expectations operator, the expectation being taken with respect to the distribution of ϵ_{t+1}, and

$$v_{2t}(y_{t-1}) = \min_{x_{2t}} E_t[w_2(y_t) + \beta_2 v_{2,t+1}(y_t) \mid x_{1t} = X_{1t}(y_{t-1}, x_{2t}, \epsilon_t)],$$

both subject to (9).

We see that the problem formulation for the dominant player is as before, except that in equilibrium he foresees how the decisions of player 1 depend on the disturbance terms. Player 1, on the other hand, knows y_t with certainty when deciding on x_{1t}, and to him there is only uncertainty with regard to future disturbances, the expected influence of which is taken into account in his value function.

The computations of equilibrium decision rules and value functions for period t will now be outlined. We assume that the value functions $v_{1,t+1}(y_t, \epsilon_{t+1})$ and $v_{2,t+1}(y_t)$ have been determined by backward induction. It is easy to see that these functions will be of the form:

$$v_{1,t+1}(y_t, \epsilon_{t+1}) = \nu_{1,t+1} + r'_{1,t+1}y_t + \tfrac{1}{2}y_t'S_{1,t+1}y_t + \xi_{t+1}\epsilon_{t+1}$$
$$+ y_t'L_{t+1}\epsilon_{t+1} + \tfrac{1}{2}\epsilon'_{t+1}W_{t+1}\epsilon_{t+1},$$
$$v_{2,t+1}(y_t) = \nu_{2,t+1} + r'_{2,t+1}y_t + \tfrac{1}{2}y_t'S_{2,t+1}y_t.$$

Consider now the problem facing player 1. His decision rule is found by solving

$$\begin{aligned} &V_{1t}(y_{t-1}, x_{2t}, \epsilon_t) \\ &\quad = \min_{x_{1t}} [w_1(y_t) + \beta_1 E_{t+1} v_{1,t+1}(y_t, \epsilon_{t+1})] \\ &\quad = \min_{x_{1t}} \{\beta_1[\nu_{1,t+1} + \tfrac{1}{2}\operatorname{trace}(W_{t+1}\Omega)] + \rho'_{1t}(Ay_{t-1} + Bx_t + c + \epsilon_t) \\ &\qquad + \tfrac{1}{2}(Ay_{t-1} + Bx_t + c + \epsilon_t)'\,\Sigma_{1t}(Ay_{t-1} + Bx_t + c + \epsilon_t)\}. \end{aligned}$$

From the first-order condition we get a decision rule of the form

$$x_{1t} = g_t y_{t-1} + \eta_t x_{2t} + u_t\epsilon_t + \gamma_t, \tag{13}$$

the coefficients of which are computed as in Theorem 1, except for the row vector u_t, which is given by

$$u_t = -[b_1'\Sigma_{1t}b_1]^{-1}\, b_1'\Sigma_{1t}.$$

Player 2, taking account of this decision rule, determines

$$v_{2t}(y_{t-1}) = \min_{x_{2t}} E_t[w_2(y_t) + \beta_2 v_{2,t+1}(y_t)],$$

subject to (9) and (13). From the first-order condition we get:

$$x_{2t} = d_{2t}y_{t-1} + \delta_{2t}, \tag{14}$$

where d_{2t} and δ_{2t} are given by Theorem 1.

Using (14) we can write (13) as

$$\begin{aligned} x_{1t} &= (g_t + \eta_t d_{2t})\,y_{t-1} + u_t\epsilon_t + (\gamma_t + \eta_t\delta_{2t}) \\ &\equiv d_{1t}y_{t-1} + u_t\epsilon_t + \delta_{1t}. \end{aligned}$$

Substituting the equilibrium decision rules, the structural equations can be written as

$$y_t = (A + BD_t)\,y_{t-1} + (B\delta_t + c) + (I + b_1u_t)\,\epsilon_t, \tag{15}$$

where $D_t = [d'_{1t}, d'_{2t}]'$, $\delta_t = (\delta_{1t}, \delta_{2t})'$, and I is the m-dimensional identity matrix. Substituting from (15) into the value functions $v_{1t}(y_{t-1}, \epsilon_t) = V_{1t}(y_{t-1}, x_{2t}, \epsilon_t \mid x_{2t} = d_{2t}y_{t-1} + \delta_{2t})$ and $v_{2t}(y_{t-1})$, and comparing coefficients, we find that the recursive relations for S_{it} and r_{it}, $i = 1, 2$, are the same as in Theorem 1, while the remaining coefficients are given by:

$$\begin{aligned}
\nu_{1t} &= [\rho_{1t} + \tfrac{1}{2}\Sigma_{1t}(B\delta_t + c)]'(B\delta_t + c) + \beta_1[\nu_{1,t+1} + \tfrac{1}{2}\operatorname{trace}(W_{t+1}\Omega)], \\
\nu_{2t} &= [\rho_{2t} + \tfrac{1}{2}\Sigma_{2t}(B\delta_t + c)]'(B\delta_t + c) + \tfrac{1}{2}\operatorname{trace}[(I + b_1u_t)'\,\Sigma_{2t}(I + b_1u_t)\,\Omega] \\
&\quad + \beta_2\nu_{2,t+1}, \\
W_t &= (I + b_1u_t)'\,\Sigma_{1t}(I + b_1u_t), \\
L_t &= (A + BD_t)'\,\Sigma_{1t}(I + b_1u_t), \\
\xi_t &= [\rho_{1t} + \Sigma_{1t}(B\delta_t + c)]'(I + b_1u_t).
\end{aligned}$$

We have thus ended up with equilibrium decision rules for a model in which there is a difference in the amount of information available to the players when making their decisions for a given time period. In models where there is no clear institutional reason why one firm is dominant or not, and in which the nondominant role makes the firm worse off than under the noncooperative solution, this possible difference in the amount of information may make the nondominant role more acceptable.

It is interesting to note that, from the point of view of the dominant player, the decision rule of the nondominant player is stochastic. Even if he knows the decision rule of his rival, he will not know for sure what the outcome will be.

For the infinite-horizon problem the stationary decision rules would be of the form

$$x_1 = gy_{-1} + \eta x_2 + \gamma + u\epsilon,$$

and

$$x_2 = d_2 y_{-1} + \delta_2 .$$

The mean of the stationary solution is the same as in Section 3, while the covariance matrix Λ for the stationary solution y^* now can be found by solving

$$\Lambda = (A + BD)\,\Lambda(A + BD)' + (I + b_1u)\,\Omega(I + b_1u)'.$$

5. Concluding Comments

In this paper we have argued that operational characteristics of economic models, and in particular stability considerations, point strongly toward an equilibrium concept for dynamic dominant-player models which implies that the players determine their best decisions depending on the current state of the system and the decisions of the other players, and rationally expecting that equilibrium decisions will be chosen in the future. This solution is called the feedback solution. Unlike two alternative and different solutions it has the property that the original plan is consistent under replanning. The difference between the solutions in this regard does not depend on the presence of uncertainty. Because of this property, the feedback solution is the only one that seems likely to be stable in the sense that decision makers groping for equilibrium decision rules will converge on these decision rules. In equilibrium, expections are self-fulfilling in the sense that the expected decision rules of the other players are actually the ones being used. One might go on by studying processes in which these equilibrium rules, if the system is stable, would be the end result of an iterative scheme of expectations formation with regard to decision rules. Using analytical methods, or, since the models are rather complex, computer simulations, one could get an idea of the robustness of the models with respect to various ways of forming such expectations, for instance, adaptive expectations. Attempts in this direction have been made in other contexts (see [6, 7]).

We have also pointed to other problems in need of further research. We have barely touched upon the infinite-horizon problem. Typically the life of the economy is not finite, and even when the horizon is finite but long, the first-period solution is usually very close to the stationary decision rules for the infinite-period problem. General conditions for existence and uniqueness of infinite-horizon feedback solutions remain to be developed.

We believe that the theory presented in this paper provides a useful framework for studies of various economic problems. This type of model, possibly with the generalization to more complex hierarchical structures, appears particularly promising for attempting to explain certain stylized facts of industry structure, and work is currently being done in that direction. The possible application to problems of public policies should also be mentioned.

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